Symmetries in quantum and classical field theories

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The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.

Henri Poincaré

Abstract

The initial chapter of the thesis provides a review of Weinberg's formalism for the derivation of quantum fields. The formalism is extended to allow for the derivation of quantum fields with more than one spin degree of freedom. It is conjectured that it may be possible to construct massive bosonic quantum field theories of any desired spin j that are consistent and unitary at all energies without the need for regulator terms by including $j + 1$ spin degrees of freedom: j, $j - 1$, down to $j - j$. The concept is then demonstrated in two subsequent chapters by the derivation of a quantum field with spin one and spin zero degrees of freedom followed the derivation of a quantum field with spin two, spin one, and spin zero degrees of freedom. Both field theories are found to be consistent and unitary at all energies without the need for regulator terms. The final two chapters are on unrelated topics. The penultimate chapter provides an explicit derivation of quantum fields for massless particles of spin one-half. In the final chapter, a derivation of the free-space Proca and Maxwell equations is provided via a consistent identification of the linear combinations of the classical fields of the $(1, 0)$ and $(0, 1)$ representations of the orthochronous Lorentz group.

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Overview

1.1 Introduction

Invariably the most unifying concept in theoretical physics over the past century has been that of symmetry. It has inspired and enabled physicist to construct ever more beautiful, ever more complete, theories of nature. In order to put the present notion into a modern context, specifically in terms of the covariance of the laws of nature under the interval-preserving linear transformations of Minkowski space, it will be of value to recall the historical developments that led to the original discovery of these transformations. We begin with Maxwell.

Electromagnetism derives from a set of empirical relations that were corrected and brought into their modern form by Maxwell [1], a man often referred to as the Newton of electromagnetism. The story of his monumental contribution is traced back to two historic readings respectively in 1855 and 1856 both under the title *On Faraday's lines of force* [2, 3]. These were followed by a publication in four parts over the years 1861 and 1862 *On physical lines of force* [4, 5] and in 1865 by *A dynamical theory of the electromagnetic field* [6]. The culmination of his work and that of others was published under the title *A treatise on electricity and magnetism* [7] in 1873. Not only did Maxwell provide a consistent set of equations for the description of electromagnetic phenomena by adding a term to Ampére's circuital law, he also transformed the geometrical ideas of Faraday into precise mathematical statements. He showed that electric and magnetic fields satisfy a wave equation with a constant speed of propagation which he found, for propagation in the hypothetical "luminiferous ether," to be in close agreement with what was at that time the experimentally established value for the speed of light [4, 5]. Faraday had previously offered an explanation for the so called Faraday effect [8], a phenomenon in which the plane of polarisation of a light beam is rotated as it passes through a magnetic field, by proposing that light is some kind of undulation in his lines of force [5]. This idea was further supported by Maxwell's analysis of the corrected electromagnetic equations. Maxwell recognised the connection between propagating electromagnetic fields and light [6], thus unifying the theories of optics and electromagnetism.

In parallel to Maxwell's triumphant advancement of mathematical physics was the development in mathematics of group theory as the unifying concept between number theory, the theory of equations, geometry, and crystallography [9–11]. Through Cayley in 1854 [12], and independently by Kronecker in 1870 [13], came the definition of an abstract group [14]. Then between 1870 and the turn of the century came the idea of continuous groups by Lie [15] and the classification of simple finite-dimensional Lie groups by Killing [16–19] and Cartan [20]. The final decade of the 19th century saw the advent of the theory of group representations by Frobenius, Schur, and Burnside [9]. The impact of these mathematical developments on the course of physics can hardly be overstated as is aptly demonstrated by the two great revolutions in theoretical physics that took place in the early 20th century through the discovery of special relativity and quantum mechanics [9, 10, 21–25].

The group of symmetries that forms the foundation of our currently best understanding of nature is the group of linear mappings of Minkowski space [26–28] to itself that preserve the interval, a pseudo-Euclidean distance [29, p. 118]. This group of symmetries is usually referred to as the Poincaré group or the inhomogeneous Lorentz group. Lorentz in [30, 31] and Lamor in [32] derived Lorentz symmetries as a set of transformations under which Maxwell's equations remain invariant [33, 34]; whereupon, Poincaré proved [35] that these transformations form a group. Maxwell's equations thus played a crucial role in the discovery of Poincaré spacetime symmetries. Furthermore, the invariance of Maxwell's equations under Lorentz transformations was an important element in the formulation of special relativity provided by Einstein in his famous 1905 paper *On the electrodynamics of moving bodies* [36]. In spite of sustained interest within the theoretical physics community in Poincaré symmetry violation, high precision experiments [37–39] to date fail to detect any notable deviations[1] for standard model matter. We therefore take this symmetry group as the Ansatz for the theoretical exploration of the present work.

Before we proceed with the derivation of quantum fields in the next chapter, some general remarks on quantum field theory are in order. We refrain from giving a detailed account of the historical developments that gave rise to quantum field theory and choose instead to focus on some specific elements concerning the underlying group structure in terms of Lie algebra. In the introduction to volume one of *The quantum theory of fields*, Weinberg shows how field theory follows (with some caveats) as the inevitable consequence of the principles of special relativity and quantum mechanics. Unfortunately for quantum field theory, however, the union of special relativity and quantum mechanics is far from complete. This is manifest in the want of a position operator in quantum field theory, raising questions about the operational meaning of the localisation of the so called in and out states. Turing again to Weinberg [40, p. 31] we read the following: "Quantum field theory deals with fields $\psi(x)$ that destroy and create particles at a spacetime point x." Similar remarks are widespread throughout the literature [41–47]; yet, the question remains: How is this to be measured? The want of a position operator also immediately highlights that the underlying algebra of quantum mechanics, the Heisenberg algebra, does not feature in quantum field theory, save perhaps in the form of equal time (anti-) commutators of canonical field variables. It was shown by Chryssomalakos and Okon in 2004 [48] that the

[1] There are some well known caveats in the case of space-inversion and time-reversal that have been observed in weak interactions. A detailed account is given by Weinberg in [40, Sec. 3.3].

algebra obtained by naïvely combining that of Poincaré with that of Heisenberg is unstable under infinitesimal perturbations of the underlying structure constants. Chryssomalakos and Okon offer a larger algebra which is stable; however, a theory of nature based upon this algebra is yet to be developed. Group theoretic considerations thus lead beyond the present formulation of quantum field theory, and one should hope that this shall be to a framework in which some currently open questions will be satisfactorily resolved.

1.2 Document outline

- Chapter one provides an introduction, a document outline, and the acknowledgements.

- Chapter two provides a review and an extension of Weinberg's formalism for the derivation of quantum fields. The treatment is generalised to allow for the construction of quantum fields with more than one spin degree of freedom.

- Chapter three provides the first application of the general formalism of chapter two by deriving a quantum field with spin one and spin zero degrees of freedom.

- Chapter four provides the second application of the general formalism of chapter two by deriving a quantum field with spin two, spin one, and spin zero degrees of freedom.

- Chapter five provides a conclusion of chapters two, three, and four.

- Chapter six provides a brief review of Weinberg's formalism for the derivation of quantum fields for massless particles of positive energy. This is exemplified by deriving quantum fields for particles of spin one-half.

- Chapter seven provides a derivation of the free-space Proca and Maxwell equations.

- Appendix A provides remarks on notation and conventions used throughout the thesis.

- Appendix B provides some explicit derivations that are omitted in the main text.

- Appendix C provides Maclaurin series expansions of some boost and rotation operators.

- Appendix D provides two published works on Elko, a Lorentz violating dark matter candidate. This material is completely peripheral to the tenor of the thesis; therefore, it is not included among the main chapters. Nonetheless, it is included here in an appendix for the sake of a more complete account of the author's doctoral research.

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2

Derivation of multi-spin quantum fields

The here adopted approach originates with Wigner's seminal paper of 1939 *On unitary representations of the inhomogeneous Lorentz group* [49] in which he extended the method of Frobenius [50–52], the method of induced representations [53–56], to find representations of the inhomogeneous Lorentz group and developed his notion of particles as representations of the inhomogeneous Lorentz group characterised by mass and spin. These ideas were further explored by many of the greatest names in the early days of particle physics such as Bargmann, Wightman, Joos, Mackey, and others [53, 54, 57–63]. Wigner's description of particles along with the cluster decomposition principle and the demand of a Poincaré invariant S-matrix forms the physical foundation of Weinberg's quantum field theoretic framework presented initially in a series of papers in the 1960s [64–66] and later, in greater detail, in volume one of *The quantum theory of fields* [40].

We begin by providing a review of the above summarised Weinberg formalism and give an extension thereof to allow for the investigation of an effect of lower spin components on the consistency and unitarity of massive bosonic quantum field theories of spin equal to or greater than one. With the space of physical states as the first element in Weinberg's construction of quantum fields, we begin by generalising the state space to include an explicit spin index; this allows for the construction of quantum fields that include more than one value of spin. It is shown that such quantum fields admit a spin-dependent phase in the definition of the metric on the space of coefficient functions. This freedom is exploited in the derivation of the propagator to yield theories that are consistent and unitary at all energies without the need for regulator terms. A general result is anticipated at the end of the present chapter in the form of a conjecture that can be summarised as follows: one may construct a massive bosonic quantum field with highest spin degree freedom j that is consistent and unitary at all energies without the introduction of regulator terms by including in the quantum field not only j, but also all lower spin degrees of freedom $j-1$, $j-2$, and so on, down to $j-j$. An outline of a yet to be formalised proof is also given.

The here developed multi-spin formalism is illustrated by two examples. It is shown in Ch. 3 that a quantum field containing spin one and spin zero degrees of freedom is consistent and unitary at all energies without the need for regulator terms. The second example is given in Ch. 4 where a quantum field containing spin two, spin one, and spin zero degrees of freedom is derived. Here, too, one obtains a theory that is consistent and unitary at all energies without the need for coutnerterms.

2.1 The inhomogeneous Lorentz group

The inhomogeneous Lorentz group is the group of linear transformations

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu}, \tag{2.1.1}$$

on the coordinate vectors, $x \equiv (x^{\mu}) \equiv (x^0, x^1, x^2, x^3)$ and $y \equiv (y^{\mu}) \equiv (y^0, y^1, y^2, y^3)$, that leaves the pseudo-Euclidean distance [29, p. 118]

$$(x-y)\cdot(x-y) \equiv (x-y)^{\mu}\, \eta_{\mu\nu}\, (x-y)^{\nu} \equiv (x-y)^{\mu}\, (x-y)_{\mu}\,, \tag{2.1.2}$$

unchanged. Here, and throughout, we adhere to the Einstein summation convention unless the contrary is explicitly stated; η is the Minkowski metric with non-zero entries given by $\eta_{00} = -\eta_{ii} = 1$, $i \in \{1,2,3\}$. The elements of the inhomogeneous Lorentz group are thus given by the set of real constant vectors a and real constant matrices Λ that satisfy the Lorentz condition

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}. \tag{2.1.3}$$

It is easy to show that the Lorentz transformations satisfy the defining properties of a group: closure, associativity, the existence of an identity element and that of an inverse. Given that the determinant of a product is the product of the determinants, the Lorentz condition immediately implies that $\det(\Lambda) = \pm 1$.

An important subgroup of the inhomogeneous Lorentz group consists of the set of transformations that are connected to the identity transformation by a continuous parameter and that preserve the direction of time. Considering that the identity element is of determinant one and that $\det(\Lambda)$ is a continuous function of the components $\Lambda^{\mu}{}_{\nu}$, it follows that this subgroup is characterised by

$$\det(\Lambda) = 1 \quad \text{and} \quad \Lambda^0{}_0 \geq 1. \tag{2.1.4}$$

It is called the inhomogeneous proper ($\det(\Lambda) = 1$) orthochronous ($\Lambda^0{}_0 \geq 1$) Lorentz group. It is also sometimes referred to as the restricted inhomogeneous Lorentz group or the restricted Poincaré group and shall hence forth be denoted by $\mathcal{P}^{\uparrow}_{+}$, where $^{\uparrow}$ denotes the preservation of the sign of the time coordinate and $_{+}$ denotes the positive sign of the determinant. Similarly the homogeneous proper orthochronous Lorentz group or simply the proper orthochronous Lorentz group is sometimes referred to as the restricted Lorentz group. We here denote this group by $\mathcal{L}^{\uparrow}_{+}$.

Beyond the above continuous transformations, there are discrete Lorentz transformations corresponding to the symmetries of space-inversion and time-reversal. Denoting these by $\mathscr{P}$ and $\mathscr{T}$, respectively, they are give by

$$\mathscr{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathscr{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.1.5}$$

These generate a finite abelian subgroup of the Lorentz group of order 4: the group of reflections [29, p. 271]

$$\mathcal{I} \equiv \{1, \mathscr{P}, \mathscr{T}, \mathscr{P}\mathscr{T}\}. \tag{2.1.6}$$

Each element $\mathcal{I}$ (with the obvious exception of the identity) is manifestly of order two and thereby involutory. Both $\mathscr{P}$ and $\mathscr{T}$ have determinant negative one and thus are not found among the transformations of $\mathcal{L}_+^\uparrow$.

Beyond the stated physical significance of these operators, they greatly simplify the mathematical development of the Lorentz group. As Weinberg points out in [40, p. 58], any Lorentz transformation can be expressed as a proper orthochronous Lorentz transformation or as a product of a proper orthochronous Lorentz transformation and one of the reflections. A study of the full Lorentz group can therefore be conducted by exploring its proper orthochronous subgroup along with space-inversion and time-reversal. We will discuss these symmetries in further detail in the context of the state space where they will be represented either by unitary or antiunitary operators.

For the remainder of this section and the two subsections to follow we shall devote our attention to the restricted homogeneous Lorentz group. Being a Lie group, much of its structure can be deduced by looking at the elements $\Lambda^\mu{}_\nu$ near the identity. Consider

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu, \quad \text{with} \quad (\lambda^\mu{}_\nu)^2 \approx 0. \tag{2.1.7}$$

From condition (2.1.3), we then obtain

$$\begin{aligned} \eta_{\alpha\beta} &= \eta_{\mu\nu} \left(\delta^\mu{}_\alpha + \lambda^\mu{}_\alpha\right) \left(\delta^\nu{}_\beta + \lambda^\nu{}_\beta\right) \\ &= \eta_{\alpha\beta} + \lambda_{\alpha\beta} + \lambda_{\beta\alpha}. \end{aligned} \tag{2.1.8}$$

Hence, $\lambda_{\mu\nu}$ is an antisymmetric second rank tensor with six degrees of freedom corresponding to the six parameters of the homogeneous Lorentz group.

2.1.1 The little group: W

Of special interest in the next section will be a particular subgroup of the Lorentz group known as the little group [49, p. 184]. Before we proceed to define this group, we must first introduce a related concept, namely that of a standard four-momentum. Toward this objective, note that $p \cdot p$ is invariant under the action of any proper orthochronous Lorentz transformation. Furthermore, for $p \cdot p \geq 0$, the sign of p^0 is left unchanged. We may thus categorise all four-momenta in terms of the value of $p \cdot p$ and in the case of $p \cdot p \geq 0$, the sign of p^0. Any four-momentum p^μ of a given class may then be defined in terms of a standard vector k^μ of that class, along with a Lorentz boost $L(p)$, such that

$$p^\mu = L^\mu{}_\nu(p)\, k^\nu. \tag{2.1.9}$$

There are altogether six classes of four-momenta as given in [40, p. 66]. Two of these (see Tab. 2.1) are of relevance here and will thus be considered in detail in Secs. 2.1.1 and 2.1.1.

Having thus introduced the standard vector k^μ, the related notion of a little group readily follows. The little group W is a subgroup of $\mathcal{L}_+^\uparrow$ with elements $W^\mu{}_\nu$ that satisfy

$$W^\mu{}_\nu k^\nu = k^\mu, \tag{2.1.10}$$

for a given standard vector k^μ. To show that W is a subgroup of $\mathcal{L}_+^\uparrow$, note that the elements of W are also elements of $\mathcal{L}_+^\uparrow$; consequently, the binary operation of multiplication and the associativity property of $\mathcal{L}_+^\uparrow$ is inherited by W. Furthermore, W contains the identity $\delta^\mu{}_\nu$. For every element $W^\mu{}_\nu \in W$, the inverse $W_\mu{}^\nu \in \mathcal{L}_+^\uparrow$ is also an element of W, as is manifest by applying $W_\mu{}^\nu$ on both sides of (2.1.10). The only remaining property to be checked is that of closure. Consider two elements $W^\mu{}_\nu$ and $\bar{W}^\mu{}_\nu$ of W. Then, from (2.1.10), we have

$$W^\mu{}_\nu \bar{W}^\nu{}_\lambda k^\lambda = W^\mu{}_\nu k^\nu = k^\mu, \tag{2.1.11}$$

that is, $W^\mu{}_\nu \bar{W}^\nu{}_\lambda$ is also an element of W; thus, W is closed. This completes the proof that W forms a subgroup of $\mathcal{L}_+^\uparrow$.

In Secs. 2.1.1 and 2.1.1 we will seek to constrain the six free parameters of an infinitesimal Lorentz transformation to obtain the infinitesimal symmetry generators of the respective little groups corresponding to the two standard vectors given in Tab. 2.1. It will therefore prove useful to convert (2.1.10) into an expression for an infinitesimal little group transformation. Taking $W^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, where $(\omega^\mu{}_\nu)^2 \approx 0$, it follows immediately from (2.1.10) that

$$\omega^\mu{}_\nu k^\nu = 0. \tag{2.1.12}$$

Hence, an infinitesimal little group transformation $\omega^\mu{}_\nu$ must annihilate the standard vector. This requirement is equivalent to (2.1.10) because $\mathcal{L}_+^\uparrow$, and thereby every subgroup W, is a Lie group.

Defining property	Standard vector	Little group	Symmetry generators
$p^0 > 0,\ p \cdot p > 0$	$(m, 0, 0, 0)$	SO(3)	$\mathcal{J}_x,\ \mathcal{J}_x,\ \mathcal{J}_x$
$p^0 > 0,\ p \cdot p = 0$	$(\kappa, 0, 0, \kappa)$	ISO(2)	$\mathcal{J}_z,\ \mathcal{K}_x - \mathcal{J}_y,\ \mathcal{K}_y - \mathcal{J}_x$

Table 2.1: Classes of four-momenta corresponding, respectively, to a massive and a massless particle, both with positive energy. Their respective standard vectors are given along with the associated little groups. Here m and κ are positive nonzero real numbers.

The particular choice of standard vectors, as given in Tab. 2.1, is not unique; it is, however, of no physical consequence, and the choices here taken are computationally favourable. The first standard vector given in Tab. 2.1 is the four-momentum of a massive particle at rest. This is certainly the simplest and most convenient choice insofar as no consideration of any prior orientation need be made in writing down the explicit form of the boost $L(p)$ in (2.1.9). As for the second standard vector in Tab. 2.1, this is the four-vector of a massless particle of energy κ moving parallel to the z-axis. κ may be set to unity or any other positive non-zero value in whatever units one may happen to prefer. The non-uniqueness and inconsequential nature of the particular choice of standard vectors will become clear once the little group for massless particles has been derived in Sec. 2.1.1.

Massive particles

To determine the set of Lorentz transformations that constitute the little group for a massive particle of positive energy, consider

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \lambda^{\mu}{}_{\nu}, \quad \text{with} \quad (\lambda^{\mu}{}_{\nu})^2 \approx 0, \tag{2.1.13}$$

where $\Lambda^{\mu}{}_{\nu}$ is a Lorentz transformation expanded infinitesimally to first order. Applying this to the first standard vector in Tab. 2.1, given by $k^0 = m$ and $k^i = 0$, and imposing the defining property for an infinitesimal little group transformation (2.1.12) on $\lambda^{\mu}{}_{\nu}$ in (2.1.13), we obtain three constraints on the second order antisymmetric tensor $\lambda_{\mu\nu}$ leaving the following non-zero components

$$\lambda_{32} = -\lambda_{23} \equiv \theta_x, \quad \lambda_{13} = -\lambda_{31} \equiv \theta_y, \quad \text{and} \quad \lambda_{21} = -\lambda_{12} \equiv \theta_z. \tag{2.1.14}$$

The little group for a massive particle of positive energy is thus a three parameter subgroup of the Lorentz group. We may find the associated Lie algebra by computing the underlying symmetry generators and studying their properties under the Lie bracket. Taking (2.1.13), with non-zero elements in $\lambda^{\mu}{}_{\nu} = \eta^{\mu\alpha}\lambda_{\alpha\nu}$ given as per (2.1.14), we compute

$$\mathscr{J}_j = \frac{1}{i}\frac{\partial}{\partial\theta_j}\Lambda^{\mu}{}_{\nu}, \quad \text{with} \quad j \in \{x, y, z\}, \tag{2.1.15}$$

to find three infinitesimal symmetry generators: $\mathscr{J}_x$, $\mathscr{J}_y$, and $\mathscr{J}_z$. Under the Lie bracket, these satisfy

$$[\mathscr{J}_i, \mathscr{J}_j] = -i\,\epsilon_{ijk}\,\mathscr{J}_k, \tag{2.1.16}$$

where $\epsilon_{123} = -1$; the same convention is used throughout. We recognise (2.1.16) as the Lie algebra of the rotation group. The little group for massive particles of positive energy is thus the three-dimensional rotation group, SO(3), as was to be shown. We shall denote this group in terms of the associated rotation parameters by $W(\theta_x, \theta_y, \theta_z)$.

Massless particles

Again consider an infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \lambda^{\mu}{}_{\nu}, \quad \text{with} \quad (\lambda^{\mu}{}_{\nu})^2 \approx 0, \tag{2.1.17}$$

and impose the defining property of an infinitesimal little group transformation (2.1.12) upon (2.1.17) as applied to the second standard vector in Tab. 2.1, for which $k^0 = k^3 = \kappa$ and $k^1 = k^2 = 0$. We thus obtain three independent constraints on the six free parameters of the second order antisymmetric tensor $\lambda_{\mu\nu}$; this leaves the following three free parameters

$$\begin{aligned}
&\lambda_{21} = -\lambda_{12} \equiv \vartheta_1, \\
&\lambda_{10} = -\lambda_{01} = \lambda_{31} = -\lambda_{13} \equiv -\vartheta_2, \\
&\lambda_{20} = -\lambda_{02} = \lambda_{32} = -\lambda_{23} \equiv -\vartheta_3.
\end{aligned} \tag{2.1.18}$$

Again, the little group is a three parameter subgroup of the Lorentz group. To obtain the corresponding symmetry generators, we compute

$$\chi_j = \frac{1}{i}\frac{\partial}{\partial \vartheta_j}\Lambda^{\mu}{}_{\nu}, \quad \text{with} \quad j \in \{1,2,3\}, \tag{2.1.19}$$

where $\Lambda^{\mu}{}_{\nu}$ is as in (2.1.17) and $\lambda^{\mu}{}_{\nu}$ has non-zero components given by (2.1.18). The generators χ_j are found to have the following properties under the Lie bracket

$$[\chi_1, \chi_2] = i\chi_3, \quad [\chi_1, \chi_3] = -i\chi_2, \quad \text{and} \quad [\chi_2, \chi_3] = 0. \tag{2.1.20}$$

This algebra is identified as that of the group of rotations and translations in two dimension, ISO(2). This completes the derivation of the little group for massless particles of positive energy. For future reference, it is worth noting that the generators χ_j can be written in terms of the generators of the Lorentz group as

$$\chi_1 = \mathscr{J}_z, \quad \chi_2 = \mathscr{K}_x + \mathscr{J}_y \equiv A, \quad \text{and} \quad \chi_3 = \mathscr{K}_y - \mathscr{J}_x \equiv B. \tag{2.1.21}$$

Accordingly, the parameters are redefined as $\vartheta_1 \equiv \theta_z$, $\vartheta_2 \equiv \alpha$, and $\vartheta_3 \equiv \beta$. The little group for massless particles of positive energy may thus be denoted $W(\theta_z, \alpha, \beta)$.

Before proceeding to the next section, we briefly return to the earlier remark regarding the non-uniqueness in the choice of k^μ. Instead of taking the three-momentum k^i to be aligned along the z-axis, we could equally well have chosen this to be aligned along x or y. This would simply alter the expansion of χ_j in terms of the symmetry generators of the Lorentz group by a cyclic permutation of the spatial coordinate indices $\{x, y, z\}$. Such a permutation, however, would not have any physical consequences because the Lie algebra (2.1.20) would remain unaltered and thus the little group for a massless particle of positive energy would still be ISO(2).

2.2 The Lie algebra

We here derive the Lie algebra of the restricted Poincaré group and derive the transformation properties of the associated generators under the representations of the Poincaré group.

2.2.1 The continuous symmetries of the restricted Poincaré group

Let $U[\Lambda, a]$ be a unitary linear representation of $\mathcal{P}_+^\uparrow$ in accordance with the treatment of Weinberg [40, Sec. 2.2]. As follows directly from (2.1.1), $U[\Lambda, a]$ must, up to a phase,[1] satisfy the following composition rule

$$U[\bar{\Lambda}, \bar{a}]\, U[\Lambda, a] = U[\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}]. \tag{2.2.1}$$

Again, as in the previous section, we will study the structure of this representation by looking at the properties of the group elements near the identity.

Consider the following expansion of the above unitary operator in the case of an infinitesimal Poincaré transformation

$$U[1+\lambda, \epsilon] = 1 + \frac{1}{2} i\lambda_{\alpha\beta} J^{\alpha\beta} + i\epsilon_\alpha P^\alpha + \ldots, \tag{2.2.2}$$

where $\lambda_{\alpha\beta}$ and ϵ_α are infinitesimal parameters; $J^{\alpha\beta}$ and P^α are operators, independent of $\lambda_{\alpha\beta}$ and ϵ_α. We already know from (2.1.8) that λ is a completely antisymmetric second rank tensor; J may therefore be taken to be antisymmetric also. Unitarity will place yet a further restriction on the generators. Expanding again to first order, we find

$$\begin{aligned} 1 &= U^\dagger[1+\lambda, \epsilon]\, U[1+\lambda, \epsilon] \\ &= \left(1 + \frac{1}{2} i\lambda_{\rho\sigma} J^{\rho\sigma} + i\epsilon_\sigma P^\sigma\right)^\dagger \left(1 + \frac{1}{2} i\lambda_{\alpha\beta} J^{\alpha\beta} + i\epsilon_\alpha P^\alpha\right) \\ &= 1 - \frac{1}{2} i \left(\lambda_{\rho\sigma} J^{\rho\sigma}\right)^\dagger - i\left(\epsilon_\sigma P^\sigma\right)^\dagger + \frac{1}{2} i\lambda_{\alpha\beta} J^{\alpha\beta} + i\epsilon_\alpha P^\alpha. \end{aligned} \tag{2.2.3}$$

The elements $\lambda_{\mu\nu} J^{\mu\nu}$ and $\epsilon_\mu P^\mu$ are traces because all indices have been summed over. It thus follows from (2.2.3) that each element of J and P must be Hermitian:

$$J^{\mu\nu\dagger} = J^{\mu\nu} \quad \text{and} \quad P^{\mu\dagger} = P^\mu. \tag{2.2.4}$$

To establish the properties of $J^{\mu\nu}$ and P^μ under the unitary representations of the

[1] In [40, Sec. 2.7], Weinberg shows that the group multiplication law for a representation of the restricted inhomogeneous Lorentz group can be chosen only up to a sign. In [40, eq. 2.7.44] we read $U[\bar{\Lambda}, \bar{a}]\, U[\Lambda, a] = \pm U[\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}]$. Nevertheless, there is no reason a priori why the restricted inhomogeneous Lorentz group ought to be chosen as the fundamental symmetry group of nature. We may just as well consider $\mathbb{R}^{1,3} \rtimes \mathrm{SL}(2,\mathbb{C})$, the double universal covering group of the restricted inhomogeneous Lorentz group. $\mathbb{R}^{1,3} \rtimes \mathrm{SL}(2,\mathbb{C})$ has no intrinsically projective representations. We will return to the question of projective representations in Sec. 2.2.2 and again in Sec. 2.3.3.

Poincaré group, consider the product

$$U[\Lambda, a]\, U[1+\lambda, \epsilon]\, U^{-1}[\Lambda, a], \tag{2.2.5}$$

where the unitary operator $U[\Lambda, a]$ represents a Poincaré transformation that is independent from that represented by $U[1+\lambda, \epsilon]$. First note from (2.2.1) that

$$U[\Lambda^{-1}, -\Lambda^{-1}a]\, U[\Lambda, a] = U[1, 0],$$

where $U[1, 0]$ is a representation of the identity. Hence, $U^{-1}[\Lambda, a] = U[\Lambda^{-1}, -\Lambda^{-1}a]$ and we can write

$$U[\Lambda, a]\, U[1+\lambda, \epsilon]\, U^{-1}[\Lambda, a] = U[\Lambda(1+\lambda)\Lambda^{-1}, \Lambda\epsilon - \Lambda\lambda\Lambda^{-1}a]. \tag{2.2.6}$$

Expanding both sides of (2.2.6) to first order in λ and ϵ yields

$$\begin{aligned} U[\Lambda, a]\left(1 + \frac{1}{2}i\lambda_{\alpha\beta}J^{\alpha\beta} + i\epsilon_\alpha P^\alpha\right) U^{-1}[\Lambda, a] \\ = 1 + \frac{1}{2}i(\Lambda\lambda\Lambda^{-1})_{\alpha\beta}J^{\alpha\beta} + i(\Lambda\epsilon - \Lambda\lambda\Lambda^{-1}a)_\alpha P^\alpha. \end{aligned} \tag{2.2.7}$$

In order to equate the coefficients on both sides, we must first fully write out the above in index form. We begin by multiplying (2.1.3) by the inverse of $\eta_{\alpha\beta}$ to obtain

$$\delta^\rho{}_\beta = \eta^{\rho\alpha}\eta_{\alpha\beta} = \eta^{\rho\alpha}\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \Lambda_\nu{}^\rho\Lambda^\nu{}_\beta.$$

Therefore, $(\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho$; accordingly we may write the coefficient of $J^{\alpha\beta}$ in the second line of (2.2.7) as

$$(\Lambda\lambda\Lambda^{-1})_{\alpha\beta} = \Lambda_\alpha{}^\mu\lambda_{\mu\nu}(\Lambda^{-1})^\nu{}_\beta = \Lambda_\alpha{}^\mu\lambda_{\mu\nu}\Lambda_\beta{}^\nu = \lambda_{\mu\nu}\Lambda_\alpha{}^\mu\Lambda_\beta{}^\nu. \tag{2.2.8}$$

Substituting into (2.2.7) and cancelling common factors, we find

$$\begin{aligned} &\frac{1}{2}\lambda_{\alpha\beta}U[\Lambda, a]J^{\alpha\beta}U^{-1}[\Lambda, a] + \epsilon_\alpha U[\Lambda, a]P^\alpha U^{-1}[\Lambda, a] \\ &= \frac{1}{2}\lambda_{\mu\nu}\Lambda_\alpha{}^\mu\Lambda_\beta{}^\nu J^{\alpha\beta} + \epsilon_\beta\Lambda_\alpha{}^\beta P^\alpha - \frac{1}{2}(\lambda_{\mu\nu} - \lambda_{\nu\mu})\Lambda_\alpha{}^\mu\Lambda_\beta{}^\nu a^\beta P^\alpha \\ &= \frac{1}{2}\lambda_{\alpha\beta}\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta J^{\mu\nu} + \epsilon_\alpha\Lambda_\beta{}^\alpha P^\beta - \frac{1}{2}\lambda_{\alpha\beta}\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta a^\nu P^\mu + \frac{1}{2}\lambda_{\alpha\beta}\Lambda_\nu{}^\beta\Lambda_\mu{}^\alpha a^\mu P^\nu \\ &= \frac{1}{2}\lambda_{\alpha\beta}\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta\left(J^{\mu\nu} - a^\nu P^\mu + a^\mu P^\nu\right) + \epsilon_\alpha\Lambda_\beta{}^\alpha P^\beta. \end{aligned} \tag{2.2.9}$$

Equating coefficients on both sides, the transformation properties of the generators are found to read

$$U[\Lambda, a] J^{\alpha\beta} U^{-1}[\Lambda, a] = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \left(J^{\mu\nu} - a^\nu P^\mu + a^\mu P^\nu \right), \tag{2.2.10}$$

$$U[\Lambda, a] P^\alpha U^{-1}[\Lambda, a] = \Lambda_\beta{}^\alpha P^\beta. \tag{2.2.11}$$

This shows that P^μ transforms as a vector and $J^{\mu\nu}$ transforms as second order tensor under the unitary representations of $\mathcal{P}_+^\uparrow$.

We now derive the Lie algebra of the Poincaré group. Again, consider an infinitesimal Poincaré transformation, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu$ and $a^\mu = \epsilon^\mu$, with infinitesimals $\lambda^\mu{}_\nu$ and ϵ^μ that are independent of those used in the preceding derivation. Multiplying (2.2.10) by $U[\Lambda, a]$ from the right to obtain

$$U[\Lambda, a] J^{\alpha\beta} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \left(J^{\mu\nu} - a^\nu P^\mu + a^\mu P^\nu \right) U[\Lambda, a],$$

we use (2.2.2) to expand both sides to first order in λ and ϵ. This yields

$$\begin{aligned}\left(1 + \frac{1}{2} i\lambda_{\rho\sigma} J^{\rho\sigma} + i\epsilon_\rho P^\rho\right) J^{\alpha\beta} = J^{\alpha\beta} &\left(1 + \frac{1}{2} i\lambda_{\rho\sigma} J^{\rho\sigma} + i\epsilon_\rho P^\rho\right) \\ &+ \lambda_\nu{}^\beta J^{\alpha\nu} + \lambda_\mu{}^\alpha J^{\mu\beta} - \epsilon^\beta P^\alpha + \epsilon^\alpha P^\beta. \end{aligned}\tag{2.2.12}$$

Proceeding likewise for (2.2.11), we obtain

$$\begin{aligned}\left(1 + \frac{1}{2} i\lambda_{\rho\sigma} J^{\rho\sigma} + i\epsilon_\rho P^\rho\right) P^\alpha &= \left(\delta_\beta{}^\alpha + \lambda_\beta{}^\alpha\right) P^\beta \left(1 + \frac{1}{2} i\lambda_{\rho\sigma} J^{\rho\sigma} + i\epsilon_\rho P^\rho\right) \\ &= P^\alpha \left(1 + \frac{1}{2} i\lambda_{\rho\sigma} J^{\rho\sigma} + i\epsilon_\rho P^\rho\right) + \lambda_\beta{}^\alpha P^\beta. \end{aligned}\tag{2.2.13}$$

Respectively, (2.2.12) and (2.2.13) may now be rewritten in terms of the commutators

$$i\left[\frac{1}{2}\lambda_{\rho\sigma} J^{\rho\sigma} + \epsilon_\rho P^\rho, J^{\alpha\beta}\right] = \lambda_\nu{}^\beta J^{\alpha\nu} + \lambda_\mu{}^\alpha J^{\mu\beta} - \epsilon^\beta P^\alpha + \epsilon^\alpha P^\beta, \tag{2.2.14}$$

$$i\left[\frac{1}{2}\lambda_{\rho\sigma} J^{\rho\sigma} + \epsilon_\rho P^\rho, P^\alpha\right] = \lambda_\beta{}^\alpha P^\beta. \tag{2.2.15}$$

Inserting the raising operator on the right had side and relabelling yields

$$\begin{aligned} i\left[\frac{1}{2}\lambda_{\rho\sigma} J^{\rho\sigma} + \epsilon_\rho P^\rho, J^{\alpha\beta}\right] &= \lambda_{\rho\sigma}\left(\eta^{\sigma\beta} J^{\alpha\rho} + \eta^{\sigma\alpha} J^{\rho\beta}\right) - \epsilon_\rho\left(\eta^{\rho\beta} P^\alpha + \eta^{\rho\alpha} P^\beta\right), \\ i\left[\frac{1}{2}\lambda_{\rho\sigma} J^{\rho\sigma} + \epsilon_\rho P^\rho, P^\alpha\right] &= \lambda_{\rho\sigma}\left(\eta^{\sigma\alpha} P^\rho\right). \end{aligned}$$

Now from the antisymmetry of λ, the above become

$$\begin{aligned}
i\left[\frac{1}{2}\lambda_{\rho\sigma}J^{\rho\sigma}+\epsilon_\rho P^\rho, J^{\alpha\beta}\right] &= \frac{1}{2}\left(\lambda_{\rho\sigma}-\lambda_{\sigma\rho}\right)\left(\eta^{\sigma\beta}J^{\alpha\rho}+\eta^{\sigma\alpha}J^{\rho\beta}\right) \\
&\quad -\epsilon_\rho\left(\eta^{\rho\beta}P^\alpha+\eta^{\rho\alpha}P^\beta\right) \\
&= \frac{1}{2}\lambda_{\rho\sigma}\left(\eta^{\sigma\beta}J^{\alpha\rho}+\eta^{\sigma\alpha}J^{\rho\beta}-\eta^{\rho\beta}J^{\alpha\sigma}-\eta^{\rho\alpha}J^{\sigma\beta}\right) \\
&\quad -\epsilon_\rho\left(\eta^{\rho\beta}P^\alpha+\eta^{\rho\alpha}P^\beta\right),
\end{aligned}$$

$$i\left[\frac{1}{2}\lambda_{\rho\sigma}J^{\rho\sigma}+\epsilon_\rho P^\rho, P^\alpha\right] = \frac{1}{2}\lambda_{\rho\sigma}\left(\eta^{\sigma\alpha}P^\rho-\eta^{\rho\alpha}P^\sigma\right).$$

Finally, equating coefficients on both sides yields the well known commutation relations that define the Lie algebra of the Poincaré group:

$$\begin{aligned}
i\left[J^{\rho\sigma}, J^{\alpha\beta}\right] &= \eta^{\sigma\beta}J^{\alpha\rho}+\eta^{\sigma\alpha}J^{\rho\beta}-\eta^{\rho\beta}J^{\alpha\sigma}-\eta^{\rho\alpha}J^{\sigma\beta}, \\
i\left[P^\rho, J^{\alpha\beta}\right] &= -\eta^{\rho\beta}P^\alpha+\eta^{\rho\alpha}P^\beta, \\
\left[P^\rho, P^\alpha\right] &= 0.
\end{aligned}$$

For future reference, and toward a more intuitive arrangement in terms of a cyclic permutation of the indices, we rewrite the algebra to read

$$i\left[J^{\rho\sigma}, J^{\alpha\beta}\right] = -\eta^{\rho\alpha}J^{\sigma\beta}-\eta^{\alpha\sigma}J^{\beta\rho}-\eta^{\sigma\beta}J^{\rho\alpha}-\eta^{\beta\rho}J^{\alpha\sigma}, \tag{2.2.16}$$

$$i\left[P^\rho, J^{\alpha\beta}\right] = \eta^{\rho\alpha}P^\beta-\eta^{\rho\beta}P^\alpha, \tag{2.2.17}$$

$$\left[P^\rho, P^\alpha\right] = 0. \tag{2.2.18}$$

We take the liberty of making a brief parenthetic remark about the interpretation of these generators in the expansion of $U[\Lambda, a]$ given above by (2.2.1). It is a well known result from group theory that there exist no finite-dimensional unitary representations of the restricted Poincaré group because of its non-compact structure. Hence, the space of physical states, which we shall define in the next section, cannot furnish a non-trivial finite-dimensional representation of the entire restricted Poincaré group [23, p. 43–51]. There exist, however, two compact subgroups of the Poincaré group. One is the four-parameter group of spacetime translations. It can be immediately identified as a subgroup by inspection of (2.2.18). The other subgroup is the rotation group. In order to see that this too forms a subgroup, it is helpful to express the above Lie algebra in three-vector notation.

This is readily achieved via the identifications

$$\boldsymbol{P} \equiv (P_x, P_y, P_z) = \left(P^1, P^2, P^3\right), \tag{2.2.19}$$

$$\boldsymbol{J} \equiv (J_x, J_y, J_z) = \left(J^{32}, J^{13}, J^{21}\right), \tag{2.2.20}$$

$$\boldsymbol{K} \equiv (K_x, K_y, K_z) = \left(J^{10}, J^{20}, J^{30}\right), \tag{2.2.21}$$

and $P^0 = H$. Here $\boldsymbol{P}$ and P^0 are the generators of spacetime translations; $\boldsymbol{J}$ and $\boldsymbol{K}$ are the generators of rotation and boost, respectively. With these identifications, the commutators (2.2.16)–(2.2.18) read

$$[J_i, J_j] = -i\epsilon_{ijk} J_k, \tag{2.2.22}$$

$$[J_i, K_j] = -i\epsilon_{ijk} K_k, \tag{2.2.23}$$

$$[K_i, K_j] = +i\epsilon_{ijk} J_k, \tag{2.2.24}$$

$$[P_i, J_j] = +i\epsilon_{ijk} P_k, \tag{2.2.25}$$

$$[P_i, K_j] = +i\delta^i{}_j H, \tag{2.2.26}$$

$$[H, K_i] = +iP_i. \tag{2.2.27}$$

All other commutators vanish. Here the indices $i, j, k \in \{x, y, z\}$. The Kronecker delta symbol $\delta^i{}_j$ is equal to one if $i = j$; otherwise it is zero. The Levi-Civita symbol is chosen such that $\epsilon_{xyz} = -1$. It is clear by inspection of (2.2.22) that the Lie algebra of the rotation group forms a subalgebra of the Lie algebra of the Poincaré group. The same is true of the Lie algebra of the translation group. From this it follows, at least locally, that the rotation group and the group of translations both form subgroups of $\mathcal{P}_+^\uparrow$. In Sec. 2.3.1 we will use the method if induced representations to build up the representations of $\mathcal{P}_+^\uparrow$ from an irreducible one-dimensional representation of the translation group.

2.2.2 The discrete symmetries of reflections

Let $P \equiv U[\mathscr{P}, 0]$ and $T \equiv U[\mathscr{T}, 0]$ be representations of the reflections $\mathscr{P}$ and $\mathscr{T}$, respectively. In accordance with the treatment of Weinberg [40, Sec. 2.6] and the foundational work by Wigner [22], each of these operators is either unitary and linear or antiunitarity and antilinear. We now explore the properties of the above generators of the Lie algebra of the restricted Poincaré group under the action of P and T. A physical argument will then be invoked to determine that P must be unitary and linear. Likewise it will be shown that T must be antiunitarity and antilinear.

We begin by writing down the composition rule

$$P\,U[\Lambda, a]\,P^{-1} = \vartheta_5 U[\mathscr{P}\Lambda\mathscr{P}^{-1}, \mathscr{P}a], \tag{2.2.28}$$

$$T\,U[\Lambda, a]\,T^{-1} = \vartheta_6 U[\mathscr{T}\Lambda\mathscr{T}^{-1}, \mathscr{T}a], \tag{2.2.29}$$

as dictated by the multiplication properties of the underlying Minkowski space transformations. The phases ϑ_5 and ϑ_6 may depend explicitly on the Lorentz transformations on the left hand side (LHS) of the above; thus, $\vartheta_5 \equiv \vartheta_5(\mathscr{P}, \Lambda, a, \mathscr{P}^{-1})$ and likewise $\vartheta_6 \equiv \vartheta_6(\mathscr{T}, \Lambda, a, \mathscr{T}^{-1})$.

Here we again encounter the question of intrinsically projective representations. In the preceding section, where we were looking at the restricted inhomogeneous Lorentz group, the possibility of projective representations was not considered because, as is explained by Sternberg in [9, p. 159], there is no ambiguity in the choice of $\mathbb{R}^{1,3} \rtimes \mathrm{SL}(2,\mathbb{C})$ as the covering group, or in the language of Schur [67, 68], the “representation group” [9, p. 158]. Unfortunately this is not so in the case of the full inhomogeneous Lorentz group. Here there are altogether eight possible choices of non-projective double covering groups [9, p. 160]. Therefore, we must take into account the possibility of a phase in the group multiplication law.

Taking the phases to be continuous functions of the continuous parameters $\Lambda^\mu{}_\nu$ and a^μ [49, p. 169] and noting that $\vartheta_5(\mathscr{P}, \mathbb{1}, 0, \mathscr{P}^{-1}) = \vartheta_6(\mathscr{T}, \mathbb{1}, 0, \mathscr{T}^{-1}) = 1$, from (2.2.28) and (2.2.29), it follows that

$$\vartheta_5(\mathscr{P}, \omega, \epsilon, \mathscr{P}^{-1}) \approx \vartheta_6(\mathscr{T}, \omega, \epsilon, \mathscr{T}^{-1}) \approx 1, \quad \text{for} \quad \omega^2 \approx \epsilon^2 \approx 0. \tag{2.2.30}$$

The phases ϑ_5 and ϑ_6 thus will not contribute to any expansion of (2.2.28) and (2.2.29) in which Λ is taken to be infinitesimal.

With this in mind, we expand (2.2.28) on both sides to first order using (2.2.2) and (2.2.8) to obtain

$$\begin{aligned} 1 + \frac{1}{2}\omega_{\mu\nu} P i J^{\mu\nu} P^{-1} + \epsilon_\mu P i P^\mu P^{-1} &= 1 + \frac{1}{2} i \left(\mathscr{P}\omega\mathscr{P}^{-1}\right)_{\alpha\beta} J^{\alpha\beta} + i \left(\mathscr{P}\epsilon\right)_\alpha P^\alpha \\ &= 1 + \frac{1}{2} i \omega_{\mu\nu} \mathscr{P}_\alpha{}^\mu \mathscr{P}_\beta{}^\nu J^{\alpha\beta} + i\epsilon_\mu \mathscr{P}_\alpha{}^\mu P^\alpha. \end{aligned}$$

Equating coefficients then yields

$$P i J^{\mu\nu} P^{-1} = i \mathscr{P}_\alpha{}^\mu \mathscr{P}_\beta{}^\nu J^{\alpha\beta}, \tag{2.2.31}$$

$$P i P^\mu P^{-1} = i \mathscr{P}_\alpha{}^\mu P^\alpha. \tag{2.2.32}$$

Repeating this for (2.2.29), we find

$$T i J^{\mu\nu} T^{-1} = i \mathscr{T}_\alpha{}^\mu \mathscr{T}_\beta{}^\nu J^{\alpha\beta}, \tag{2.2.33}$$

$$T i P^\mu T^{-1} = i \mathscr{T}_\alpha{}^\mu P^\alpha. \tag{2.2.34}$$

Recalling the identification $P^0 \equiv H$, we find the transformation of the energy operator from (2.2.32) and (2.2.34) is given by

$$P i H P^{-1} = +iH, \tag{2.2.35}$$

$$T i H T^{-1} = -iH. \tag{2.2.36}$$

Given that we should not wish to have states of negative energy, and furthermore that any symmetry operator is a map from the state space to itself [40, Sec. 2.2], we must not allow P or T to map the energy operator to its negative. Consequently, the action of parity on the state space must be unitary and linear whereas that of time-reversal must be antiunitary and antilinear. With this we may cancel the factor of i in (2.2.31)–(2.2.34) to give

$$PJ^{\mu\nu}P^{-1} = +\mathscr{P}_{\alpha}{}^{\mu}\mathscr{P}_{\beta}{}^{\nu}J^{\alpha\beta}, \tag{2.2.37}$$

$$PP^{\mu}P^{-1} = +\mathscr{P}_{\alpha}{}^{\mu}P^{\alpha}, \tag{2.2.38}$$

$$TJ^{\mu\nu}T^{-1} = -\mathscr{T}_{\alpha}{}^{\mu}\mathscr{T}_{\beta}{}^{\nu}J^{\alpha\beta}, \tag{2.2.39}$$

$$TP^{\mu}T^{-1} = -\mathscr{T}_{\alpha}{}^{\mu}P^{\alpha}. \tag{2.2.40}$$

In three-vector notation, these read

$$P\boldsymbol{J}P^{-1} = +\boldsymbol{J}, \qquad T\boldsymbol{J}T^{-1} = -\boldsymbol{J}, \tag{2.2.41}$$

$$P\boldsymbol{K}P^{-1} = -\boldsymbol{K}, \qquad T\boldsymbol{K}T^{-1} = +\boldsymbol{K}, \tag{2.2.42}$$

$$P\boldsymbol{P}P^{-1} = -\boldsymbol{P}, \qquad T\boldsymbol{P}T^{-1} = -\boldsymbol{P}, \tag{2.2.43}$$

$$PHP^{-1} = +H, \qquad THT^{-1} = +H. \tag{2.2.44}$$

Consistent with expectation, this shows that the generators of angular momentum transform as pseudovectors under space-inversion and as vectors under time-reversal. The generators of boost, however, transform as vectors under space-inversion and as pseudovectors under time-reversal. The transformation properties of the generators of spacetime translation also hold no surprises.

2.3 The state space

In the present section, we shall endeavour to define and develop the properties of the space of physical states. We closely follow the treatment of Wigner [23, p. 43–51] and of Weinberg, as found in [64] and [40, Ch. 2]. Also, the lecture notes on *Symmetries and groups* by Osborn [69] proved to be helpful. Our first step will be to derive the transformation properties of the one particle states under the action of the unitary representations of the restricted Poincaré group. We will begin with a unitary representation of the translation group on a vector space labelled by one continuous parameter, the four-momentum of the one particle state, and one discrete variable to account for degeneracies. A representation of the full ten parameter group $\mathcal{P}_{+}^{\uparrow}$ is then built up via the method of induced representations. Following this exploration of the continuous symmetries, we will turn to the discrete symmetries by first looking at CPT. Thereafter, the properties of the one particle states under the action of charge-conjugation and the reflections of space-inversion and time-reversal will be explored.

2.3.1 The method of induced representations

The method of induced representations is a procedure whereby to obtain representations of a group from the representations of an invariant subgroup. It was first developed by Frobenius [50] in the context of compact groups and later heuristically extended by Wigner [23] to obtain representations of the non-compact Poincaré group. A mathematically rigorous treatment was provided by Mackey [53, 54, 63]. Wigner refers to Mautner [70–73] and to von Neumann [74] for mathematical detail.

In order to apply the method of induced representations, we must first identify an appropriate subgroup and find a representation of this subgroup. By inspection of the Lie algebra of the restricted Poincaré group, (2.2.22)–(2.2.27), the four generators of spacetime translation P^μ form an invariant abelian subalgebra; the corresponding group, the group of spacetime translations, forms an invariant abelian subgroup of the Poincaré group. Furthermore, there exists [74–77] an irreducible one-dimensional representation [23] of the translation group given by

$$U[a] \equiv U[\mathbb{1}, a] = \mathrm{e}^{iP\cdot a}, \tag{2.3.1}$$

where $P \equiv (P^\mu)$, the generators of spacetime translations; $a \equiv (a^\mu)$, the associated parameters. Given that $[P^\mu, P^\nu] = 0$, we may introduce state vectors $|p;\lambda\rangle$, where p is four-momentum and λ is a yet to be determined degeneracy index, such that

$$P^\mu|p;\lambda\rangle = p^\mu|p;\lambda\rangle. \tag{2.3.2}$$

Consequently, these vectors transform under translations according to

$$U[a]|p;\lambda\rangle = \mathrm{e}^{ip\cdot x}\,|p;\lambda\rangle. \tag{2.3.3}$$

It is necessary to include a degeneracy label because there may exist many state vectors that exhibit the same transformation property under translations. As noted by Weinberg [40, p. 63], there are cases, such as in the description of several unbound particles, for which the degeneracy of the here defined states would be characterised by both continuous and discrete labels. In accordance with the treatment of Wigner and that of Weinberg, we here confine to the case where λ is strictly discrete.

It is easy to confirm that the vector space with elements $|p;\lambda\rangle$, as defined in (2.3.2) and (2.3.3), furnishes a representation of the translation group. Applying a second translation operator $U[\bar{a}]$ to (2.3.3), we obtain

$$U[\bar{a}]U[a]|p;\lambda\rangle = \mathrm{e}^{ip\cdot x}\,U[\bar{a}]|p;\lambda\rangle = \mathrm{e}^{i(\bar{a}+a)}\,|p;\lambda\rangle = U[\bar{a}+a]|p;\lambda\rangle. \tag{2.3.4}$$

Thus, along with a completeness relation, to be given shortly, this implies $U[\bar{a}]U[a] = U[\bar{a}+a]$, which is nothing but the composition rule for the translation subgroup of the Poincaré group. The latter is trivially obtained from the composition rule of $\mathcal{P}_+^\uparrow$, as given in (2.2.11).

Before we further proceed, a nice simplification can be achieved by a choice of basis for the states $|p;\lambda\rangle$ in terms of the states $|k;\lambda\rangle$, where k is a standard vector. Consider the state $U[L(p)]|k;\lambda\rangle$, where $U[L(p)] \equiv U[L(p), 0]$ and $L(p)$ is the standard boost defined by $p^\mu = L(p)^\mu{}_\nu k^\nu$, in accordance with the treatment of Sec. 2.1.1. Recalling the transformation property of P^μ given in (2.2.11) and using (2.3.2), we obtain

$$P^\mu U[L(p)]|k;\lambda\rangle = L(p)^\mu{}_\nu k^\nu U[L(p)]|k;\lambda\rangle = p^\mu U[L(p)]|k;\lambda\rangle. \tag{2.3.5}$$

Under translations, therefore, these states transform as

$$U[a]U[L(p)]|k;\lambda\rangle = \mathrm{e}^{ip\cdot a}\, U[L(p)]|k;\lambda\rangle, \tag{2.3.6}$$

and thus satisfy the defining properties of $|p;\lambda\rangle$ as introduced above in (2.3.2) and (2.3.3). We are thus at liberty to make the identification

$$|p;\lambda\rangle \equiv \sqrt{k^0/p^0}\, U[L(p)]|k;\lambda\rangle, \tag{2.3.7}$$

where the normalisation factor, $\sqrt{k^0/p^0}$, has been chosen such that [40, p. 67]

$$\langle p';\lambda'|p;\lambda\rangle = \delta^3(\boldsymbol{p}'-\boldsymbol{p})\,\delta_{\lambda',\lambda}. \tag{2.3.8}$$

The completeness relation then immediately follows. From (2.3.8), we have

$$\begin{aligned}\int d^3p' \sum_{\lambda'} \langle p'';\lambda''|p';\lambda'\rangle\langle p';\lambda'|p;\lambda\rangle &= \int d^3p' \sum_{\lambda'} \delta^3(\boldsymbol{p}''-\boldsymbol{p}')\,\delta_{\lambda'',\lambda'}\delta^3(\boldsymbol{p}'-\boldsymbol{p})\,\delta_{\lambda',\lambda}\\ &= \delta^3(\boldsymbol{p}''-\boldsymbol{p})\,\delta_{\lambda'',\lambda}\\ &= \langle p'';\lambda''|p;\lambda\rangle,\end{aligned}$$

yielding, as desired,

$$\int d^3p \sum_{\lambda} |p;\lambda\rangle\langle p;\lambda| = 1. \tag{2.3.9}$$

With this, the earlier claim, that the above introduced basis furnishes a one-dimensional representation of the translation subgroup of the restricted Poincaré group, is fully substantiated.

We now proceed with the method of induced representations to derive a representation of the full ten parameter group $\mathcal{P}_+^\uparrow$. Consider the state $U[W]|k;\lambda\rangle$, where W is an element of the little group, defined with respect to k. From (2.2.11), we have

$$P^\mu U[W]|k;\lambda\rangle = W^\mu{}_\nu k^\nu U[W]|k;\lambda\rangle = k^\mu U[W]|k;\lambda\rangle. \tag{2.3.10}$$

Hence, the unitary operator $U[W]$ induces a transformation among the vectors $|k;\lambda\rangle$ which leaves the momentum eigenvalue unchanged. Therefore, $U[W]|k;\lambda\rangle$ is given by a linear combination [40, p. 64]

$$U[W]|k;\lambda\rangle = \sum_{\lambda'} |k;\lambda'\rangle D^k_{\lambda'\lambda}[W], \tag{2.3.11}$$

where the coefficients $D^k_{\lambda'\lambda}[W]$ may have functional dependence on W, λ, and k [23, p.

46]. Applying a second transformation $U[\bar{W}]$, we find

$$U[\bar{W}]U[W]|k;\lambda\rangle = \sum_{\lambda'} U[\bar{W}]|k;\lambda'\rangle D^k_{\lambda'\lambda}[W] = \sum_{\lambda'\lambda''} |k;\lambda''\rangle D^k_{\lambda''\lambda'}[\bar{W}] D^k_{\lambda'\lambda}[W].$$

Conversely, applying $U[\bar{W}W]$ upon $|k;\lambda\rangle$, we obtain

$$U[\bar{W}W]|k;\lambda\rangle = \sum_{\lambda''} |k;\lambda''\rangle D^k_{\lambda''\lambda}[\bar{W}W].$$

From the composition rule (2.2.1), we have $U[\bar{W}]U[W] = U[\bar{W}W]$; therefore,

$$\sum_{\lambda'\lambda''} |k;\lambda''\rangle D^k_{\lambda''\lambda'}[\bar{W}] D^k_{\lambda'\lambda}[W] = \sum_{\lambda''} |k;\lambda''\rangle D^k_{\lambda''\lambda}[\bar{W}W]. \tag{2.3.12}$$

Finally, invoking the completeness relation (2.3.9), we obtain

$$\sum_{\lambda'} D^k_{\lambda''\lambda'}[\bar{W}] D^k_{\lambda'\lambda}[W] = D^k_{\lambda''\lambda}[\bar{W}W], \tag{2.3.13}$$

that is, the coefficients $D^k_{\lambda'\lambda}[W]$ furnish a finite-dimensional representation of the little group on a subspace defined by the vectors $|k;\lambda\rangle$.

We now return to the state $|p;\lambda\rangle$ defined in (2.3.7). In particular, we shall seek to express the state $U[\Lambda,a]|p;\lambda\rangle$ in terms of the above finite-dimensional representation of the little group. We begin by taking a closer look at the state $U[\Lambda]|p;\lambda\rangle$. This may be expressed as

$$\begin{aligned} U[\Lambda]\,|p;\lambda\rangle &= \sqrt{k^0/p^0}\, U[\Lambda]\, U[L(p)]\,|k;\lambda\rangle \\ &= \sqrt{k^0/p^0}\, U[L(\Lambda p)] U[L^{-1}(\Lambda p)\,\Lambda\, L(p)]\,|k;\lambda\rangle \\ &= \sqrt{k^0/p^0} \int d^3k' \sum_{\lambda'} U[L(\Lambda p)]\,|k';\lambda'\rangle\langle k';\lambda'|\, U[L^{-1}(\Lambda p)\,\Lambda\, L(p)]\,|k;\lambda\rangle. \end{aligned} \tag{2.3.14}$$

The purpose of inserting the identity $U[L(\Lambda p)]U[L^{-1}(\Lambda p)]$ was to isolate the transformation $U[L^{-1}(\Lambda p)\,\Lambda\, L(p)]$. The succession of Lorentz transformations $L^{-1}(\Lambda p)\,\Lambda\, L(p)$, when applied to the standard vector k, yields

$$L^{-1}(\Lambda p)\,\Lambda\, L(p)\, k = L^{-1}(\Lambda p)\,\Lambda p = k.$$

The standard vector is left unchanged; therefore, we concluded that $L^{-1}(\Lambda p)\,\Lambda\, L(p)$ is an element of the little group. In agreement with [40, p. 65] we choose to denote this by

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p)\,\Lambda\, L(p). \tag{2.3.15}$$

Substituting this into (2.3.14) and invoking (2.3.14), we can use the orthonormality relation (2.3.8) to obtain

$$\begin{aligned}
U[\Lambda]\,|p;\lambda\rangle &= \sqrt{k^0/p^0}\int d^3k' \sum_{\lambda'} U[L(\Lambda p)]\,|k';\lambda'\rangle\langle k';\lambda'|\,U[W(\Lambda,p)]\,|k;\lambda\rangle \\
&= \sqrt{k^0/p^0}\int d^3k' \sum_{\lambda'\lambda''} U[L(\Lambda p)]\,|k';\lambda'\rangle\langle k';\lambda'|k;\lambda''\rangle D^k_{\lambda''\lambda}[W(\Lambda,p)] \\
&= \sqrt{k^0/p^0}\int d^3k' \sum_{\lambda'\lambda''} U[L(\Lambda p)]\,|k';\lambda'\rangle\delta^3(\boldsymbol{k}'-\boldsymbol{k})\delta_{\lambda'\lambda''} D^k_{\lambda''\lambda}[W(\Lambda,p)] \\
&= \sqrt{k^0/p^0}\sum_{\lambda'} U[L(\Lambda p)]\,|k;\lambda'\rangle D^k_{\lambda'\lambda}[W(\Lambda,p)] \\
&= \sqrt{(\Lambda p)^0/p^0}\sum_{\lambda'} |\Lambda p;\lambda'\rangle D^k_{\lambda'\lambda}[W(\Lambda,p)]. \qquad (2.3.16)
\end{aligned}$$

We have thus derived the transformation properties of the singe particle states under the unitary representations of $\mathcal{L}^{\uparrow}_{+}$.

The transformation properties of the single particle states under $\mathcal{P}^{\uparrow}_{+}$ are now readily at hand. Making appropriate identifications in (2.2.1), we trivially obtain the decomposition

$$U[\Lambda,a] = U[1,a]\,U[\Lambda]. \qquad (2.3.17)$$

Applying this to the state $|p;\lambda\rangle$, we obtain the following transformation property under the action of the unitary representations of the full ten parameter restricted Poincaré group:

$$\begin{aligned}
U[\Lambda,a]\,|p;\lambda\rangle &= U[1,a]\,U[\Lambda]|p;\lambda\rangle \\
&= \sqrt{(\Lambda p)^0/p^0}\sum_{\lambda'} U[1,a]\,|\Lambda p;\lambda'\rangle D^k_{\lambda'\lambda}[W(\Lambda,p)] \\
&= \sqrt{(\Lambda p)^0/p^0}\sum_{\lambda'} \mathrm{e}^{iP\cdot a}\,|\Lambda p;\lambda'\rangle D^k_{\lambda'\lambda}[W(\Lambda,p)] \\
&= \sqrt{(\Lambda p)^0/p^0}\,\mathrm{e}^{i\Lambda p\cdot a}\sum_{\lambda'} |\Lambda p;\lambda'\rangle D^k_{\lambda'\lambda}[W(\Lambda,p)]. \qquad (2.3.18)
\end{aligned}$$

In order to verify that we have indeed derived a representation of $\mathcal{P}_+^\uparrow$ on the above defined vector space, we must show that the composition rule (2.2.1) is satisfied. Applying a second Poincaré transformation $U[\bar{\Lambda}, \bar{a}]$ to (2.3.18), we obtain

$$
\begin{aligned}
&U[\bar{\Lambda}, \bar{a}]\, U[\Lambda, a]\, |p; \lambda\rangle \\
&= \sqrt{\frac{(\Lambda p)^0}{p^0}}\, \mathrm{e}^{i\Lambda p\cdot a} \sum_{\lambda'} U[\bar{\Lambda}, \bar{a}] |\Lambda p; \lambda'\rangle D^k_{\lambda'\lambda}[W(\Lambda, p)] \\
&= \sqrt{\frac{(\Lambda p)^0}{p^0}} \sqrt{\frac{(\bar{\Lambda}\Lambda p)^0}{(\Lambda p)^0}}\, \mathrm{e}^{i\Lambda p\cdot a}\, \mathrm{e}^{i\bar{\Lambda}\Lambda p\cdot \bar{a}} \sum_{\lambda'\lambda''} |\bar{\Lambda}\Lambda p; \lambda''\rangle D^k_{\lambda''\lambda'}[W(\bar{\Lambda}, \Lambda p)] D^k_{\lambda'\lambda}[W(\Lambda, p)] \\
&= \sqrt{\frac{(\bar{\Lambda}\Lambda p)^0}{p^0}}\, \mathrm{e}^{i\Lambda p\cdot a + i\bar{\Lambda}\Lambda p\cdot\bar{a}} \sum_{\lambda''} |\bar{\Lambda}\Lambda p; \lambda''\rangle D^k_{\lambda''\lambda}[W(\bar{\Lambda}, \Lambda p) W(\Lambda, p)]. \qquad (2.3.19)
\end{aligned}
$$

Looking at the argument of the coefficient matrix, and using the definition of $W(\Lambda, p)$ given above in (2.3.15), we have

$$
\begin{aligned}
W(\bar{\Lambda}, \Lambda p) W(\Lambda, p) &= L^{-1}(\bar{\Lambda}\Lambda p)\bar{\Lambda} L(\Lambda p) L^{-1}(\Lambda p)\Lambda L(p) \\
&= L^{-1}(\bar{\Lambda}\Lambda p)\bar{\Lambda}\Lambda L(p) \\
&= W(\bar{\Lambda}\Lambda, p).
\end{aligned}
$$

With this, (2.3.19) becomes

$$
U[\bar{\Lambda}, \bar{a}]\, U[\Lambda, a]\, |p; \lambda\rangle = \sqrt{\frac{(\bar{\Lambda}\Lambda p)^0}{p^0}}\, \mathrm{e}^{i\Lambda p\cdot a + i\bar{\Lambda}\Lambda p\cdot\bar{a}} \sum_{\lambda'} |\bar{\Lambda}\Lambda p; \lambda'\rangle D^k_{\lambda'\lambda}[W(\bar{\Lambda}\Lambda, p)]. \tag{2.3.20}
$$

It follows from the Lorentz invariance of the scalar product that

$$
\Lambda p \cdot a = \bar{\Lambda}\Lambda p \cdot \bar{\Lambda} a.
$$

Using this identity in (2.3.20), we obtain

$$
\begin{aligned}
U[\bar{\Lambda}, \bar{a}]\, U[\Lambda, a]\, |p; \lambda\rangle &= \sqrt{\frac{(\bar{\Lambda}\Lambda p)^0}{p^0}}\, \mathrm{e}^{i\bar{\Lambda}\Lambda p\cdot(\bar{\Lambda}a + \bar{a})} \sum_{\lambda'} |\bar{\Lambda}\Lambda p; \lambda'\rangle D^k_{\lambda'\lambda}[W(\bar{\Lambda}\Lambda, p)] \\
&= U[\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}] |p; \lambda\rangle. \qquad (2.3.21)
\end{aligned}
$$

The transformation property of the one particle state, given in (2.3.18), is found to be consistent with the composition rule

$$
U[\bar{\Lambda}, \bar{a}]\, U[\Lambda, a] = U[\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}],
$$

as given in (2.2.1). We have thus obtained a representation of the restricted Poincaré group from a representation of the translation subgroup via the method of induced representations. In deriving (2.3.18), we have reduced the task of finding unitary representations of $\mathcal{P}_+^\uparrow$ to that of determining the representations of the little group transformation $W(\Lambda, p)$. In the next two sections we address this remaining question for the two physical scenarios considered in Sec. 2.1.1: first the case of a massive and then that of a massless particle of positive energy.

Massive particles

It was shown in Sec. 2.1.1 that the little group for a massive particle of positive energy is the three-dimensional rotation group SO(3); consequently, $W(\Lambda, p)$ must be a rotation, commonly called the "Wigner rotation" [75, 78]. This should come as no surprise given the explicit form of the relevant standard vector, $k = (m, 0, 0, 0)$, as per Tab. 2.1. Looking at [40, p. 69], we find Weinberg evaluates the Wigner rotation for $\Lambda = \mathscr{R}$, an arbitrary three-dimensional rotation, and shows that $W(\mathscr{R}, p) = \mathscr{R}$. He remarks that this result is significant because it indicates that the states $|p; \lambda\rangle$ have the same transformation properties under rotations as do those of non-relativistic quantum mechanics. We show in App. B.1 that when Λ is an arbitrary boost, $W(\Lambda, p)$ is a rotation, namely the rotation involved in the composition of two boosts into a single boost and a rotation. In the special case in which Λ and $L(p)$ are colinear, $W(\Lambda, p)$ is the identity.

The task of finding representations of the Wigner rotation is thus no different to that of finding representations of the little group for massive particles of positive energy; that is to say, the task of finding the coefficients $D^k_{\lambda'\lambda}[W]$ is simply that of finding a representation of SO(3) on a subspace defined by the elements $|k; \lambda\rangle$. We now show that it is possible to choose a basis such that the coefficients take the form of the standard angular momentum matrices [40, p. 68]. This will follow naturally from an exploration of the degeneracy index λ.

First recall $|p; \lambda\rangle \equiv \sqrt{k^0/p^0}\, U[L(p)]|k; \lambda\rangle$, where the states $|k; \lambda\rangle$ form a vector subspace under the unitary representations of the rotation group. Given that the vectors $|k; \lambda\rangle$ are eigenvectors of P^0 and,[2] furthermore, P^0 commutes with the generators of the rotation group, it follows that the degeneracy index λ must include the eigenvalues of $\boldsymbol{J}^2 \equiv (J_x^2 + J_y^2 + J_z^2)$ and J_i, where J_i may be any one of the rotation generators, typically chosen to be J_z. The states at rest may thus be relabelled in terms of the eigenvalues of these three commuting operators, such that

$$P^0|k; \sigma, s, n\rangle = m\,|k; \sigma, s, n\rangle, \tag{2.3.22}$$

$$J_z|k; \sigma, s, n\rangle = \sigma\,|k; \sigma, s, n\rangle, \tag{2.3.23}$$

$$\boldsymbol{J}^2|k; \sigma, s, n\rangle = s\,(s+1)\,|k; \sigma, s, n\rangle. \tag{2.3.24}$$

Here m is a continuous parameter, the mass of the particle; s takes on integer and half integer values, the intrinsic angular momentum of the particle; spin projection σ is related

[2] The rest vectors are of course eigenvectors also of $\boldsymbol{P}$, but this is of no consequence to the present argument because $\boldsymbol{P}|k; \lambda\rangle = 0$.

to s and takes on the values $\sigma \in \{s, s-1, \ldots, 1-s, -s\}$. The index n denotes any remaining degeneracy.

We now show that the above basis admits an expansion of the rotation generators to give the components standard angular momentum matrices. Suppressing for the time being the indices k and n, we obtain the following matrix elements for the operators J_z and $\boldsymbol{J}^2$:

$$\langle \sigma', s | J_z | \sigma, s \rangle = \sigma \, \delta_{\sigma' \sigma}, \tag{2.3.25}$$

$$\langle \sigma', s | \boldsymbol{J}^2 | \sigma, s \rangle = s \, (s+1) \, \delta_{\sigma' \sigma}. \tag{2.3.26}$$

The expansion of the remaining angular momentum operators is found in the standard fashion [79, Sec. 3.5], by first considering the linear combination $J_x \pm i J_y$. Applying this to $|\sigma, s\rangle$, we find

$$\begin{aligned} J_z \left(J_x \pm i J_y \right) |\sigma, s\rangle &= \left(J_z J_x \pm i J_z J_y \right) |\sigma, s\rangle \\ &= \left(J_x J_z + i J_y \pm i \left(J_y J_z - i J_x \right) \right) |\sigma, s\rangle \\ &= \left(J_x \pm i J_y \right) \left(J_z \pm 1 \right) |\sigma, s\rangle \\ &= \left(\sigma \pm 1 \right) \left(J_x \pm i J_y \right) |\sigma, s\rangle . \end{aligned}$$

This may be written as

$$\left(J_x \pm i J_y \right) |\sigma, s\rangle = A_{\pm} |\sigma \pm 1, s\rangle, \tag{2.3.27}$$

where $A_{\pm}$ is a normalisation constant. Its magnitude squared is determined by

$$\begin{aligned} |A_{\pm}|^2 &= \langle \sigma, s | \left(J_x \pm i J_y \right)^{\dagger} \left(J_x \pm i J_y \right) |\sigma, s\rangle \\ &= \langle \sigma, s | \left(J^2 - J_z^2 \mp J_z \right) |\sigma, s\rangle \\ &= s \, (s+1) - \sigma^2 \mp \sigma \\ &= s^2 + s \pm s\sigma \mp \sigma s - \sigma^2 \mp \sigma \\ &= (s \mp \sigma) \, (s \pm \sigma + 1) \, . \end{aligned} \tag{2.3.28}$$

Therefore, up to a phase which is chosen conventionally to be unity, the matrix elements of the linear combination $J_x \pm i J_y$ read

$$\langle \sigma', s | \left(J_x \pm i J_y \right) |\sigma, s\rangle = \sqrt{(s \mp \sigma) \, (s \pm \sigma + 1)} \; \delta_{\sigma' \sigma \pm 1}. \tag{2.3.29}$$

Of course $J_x + i J_y$ and $J_x - i J_y$ are the quantum mechanical raising and lowering operators, respectively. For the purposes of the present development (2.3.29) allows us to express J_x and J_y in the basis $|\sigma, s\rangle$.

The complete set of angular momentum generators thus reads

$$\begin{aligned}\left(J_x^{(s)}\right)_{\sigma'\sigma} &= \frac{1}{2}\sqrt{(s-\sigma)(s+\sigma+1)}\ \delta_{\sigma'\sigma+1} + \frac{1}{2}\sqrt{(s+\sigma)(s-\sigma+1)}\ \delta_{\sigma'\sigma-1},\\ \left(J_y^{(s)}\right)_{\sigma'\sigma} &= \frac{1}{2i}\sqrt{(s-\sigma)(s+\sigma+1)}\ \delta_{\sigma'\sigma+1} - \frac{1}{2i}\sqrt{(s+\sigma)(s-\sigma+1)}\ \delta_{\sigma'\sigma-1},\\ \left(J_z^{(s)}\right)_{\sigma'\sigma} &= \sigma\,\delta_{\sigma'\sigma},\end{aligned} \tag{2.3.30}$$

where $(J_i^{(s)})_{\sigma'\sigma} \equiv \langle\sigma', s|J_i|\sigma, s\rangle$.

The coefficients (2.3.18) are now readily at hand. Rewriting this in terms of the above derived basis, we have

$$\begin{aligned}U[\Lambda, a]|p;\sigma,s,n\rangle = \sqrt{k^0/p^0}\,\mathrm{e}^{i\Lambda p\cdot a}\int d^3k' \sum_{\sigma',s',n'} U[L(\Lambda p)]|k';\sigma',s',n'\rangle\\ \times\langle k';\sigma',s',n'|U[W(\Lambda,p)]|k;\sigma,s,n\rangle.\end{aligned} \tag{2.3.31}$$

For simplicity, consider the case where $W(\Lambda,p) = \mathscr{R}_z$, a rotation about the z-axis. Then, from (2.2.2), we can expand $U[\mathscr{R}_z]$ as

$$U[\mathscr{R}_z] = 1 + i\theta_z J_z + \dots\,. \tag{2.3.32}$$

The matrix to be determined thus becomes

$$\begin{aligned}\langle k';\sigma',s',n'|U[R_z]|k;\sigma,s,n\rangle &= \langle k';\sigma',s',n'|(1+i\theta_z J_z+\dots)|k;\sigma,s,n\rangle\\ &= \langle k';\sigma',s',n'|(1+i\theta_z\sigma+\dots)|k;\sigma,s,n\rangle\\ &= (1+i\theta_z\sigma+\dots)\delta_{\sigma'\sigma}\delta^3(\boldsymbol{k}'-\boldsymbol{k})\delta_{s's}\delta_{n'n}\\ &= (\delta_{\sigma'\sigma}+i\theta_z(J_z^{(s)})_{\sigma'\sigma}+\dots)\delta^3(\boldsymbol{k}'-\boldsymbol{k})\delta_{s's}\delta_{n'n}\\ &= (\delta_{\sigma'\sigma}+i\theta_z(J_z^{(s)})_{\sigma'\sigma}+\dots)\delta^3(\boldsymbol{k}'-\boldsymbol{k})\delta_{s's}\delta_{n'n}.\end{aligned}$$

The coefficient matrix is thus given by the matrix elements of the corresponding rotation expanded in terms of the above defined basis vectors; that is,

$$D^{(s)}_{\sigma'\sigma}[R_z] = \delta_{\sigma'\sigma} + i\theta_z(J_z^{(s)})_{\sigma'\sigma} + \dots, \tag{2.3.33}$$

with the obvious generalization for an arbitrary rotation. Substitution into (2.3.31) thus yields

$$\begin{aligned}U[\Lambda,a]|p;\sigma,s,n\rangle &= \sqrt{k^0/p^0}\,\mathrm{e}^{i\Lambda p\cdot a}\sum_{\sigma'} U[L(\Lambda p)]|k;\sigma',s,n\rangle D^{(s)}_{\sigma'\sigma}[W(\Lambda,p)]\\ &= \sqrt{(\Lambda p)^0/p^0}\,\mathrm{e}^{i\Lambda p\cdot a}\sum_{\sigma'}|\Lambda p;\sigma',s,n\rangle D^{(s)}_{\sigma'\sigma}[W(\Lambda,p)].\end{aligned} \tag{2.3.34}$$

With this, the transformation properties of the single particle states under the action of the unitary representations of the restricted Poincaré group is uniquely defined.

Before we proceed to the case of a massless particle of positive energy, some remarks on the labelling of the states at arbitrary momentum and about Casimir operators [80] will be in order. A Casimir operator is defined with respect to a given Lie group as one that commutes with all the elements of that Lie group [81, p. 103]. As can be readily found in the literature [82, 83], the Poincaré group has two Casimir operators: $C_1 \equiv P^\mu P_\mu$ and $C_2 \equiv W^\mu W_\mu$, where W^μ is the Pauli-Lubański pseudovector [84] given by

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} J_{\nu\alpha} P_\beta. \tag{2.3.35}$$

The completely antisymmetric Levi-Civita symbol $\epsilon^{\mu\nu\alpha\beta}$ is defined in App. A. The first Casimir operator, when applied to the state $|p;\sigma,s,n\rangle$, returns the eigenvalue m^2. To evaluate $C_2|p;\sigma,s,n\rangle$, we use (2.3.7), which now reads

$$|p;\sigma,s,n\rangle = \sqrt{m/p^0}\, U[L(p)]|k;\sigma,s,n\rangle, \tag{2.3.36}$$

along with the Lorentz invariance of $W^\mu W_\mu$ we obtain

$$\begin{aligned} W^\mu W_\mu |p;\sigma,s,n\rangle &= \sqrt{m/p^0}\, U[L(p)] W^\mu W_\mu |k;\sigma,s,n\rangle \\ &= \sqrt{m/p^0}\, U[L(p)] \frac{1}{4}\epsilon^{\mu\nu\alpha\beta} J_{\nu\alpha} P_\beta \epsilon_{\mu\delta\rho\kappa} J^{\delta\rho} P^\kappa |k;\sigma,s,n\rangle \\ &= \sqrt{m/p^0}\, U[L(p)] \frac{m^2}{4}\epsilon^{\mu\nu\alpha 0} J_{\nu\alpha} \epsilon_{\mu\delta\rho 0} J^{\delta\rho} |k;\sigma,s,n\rangle \\ &= \sqrt{m/p^0}\, U[L(p)] \frac{m^2}{4}\epsilon^{ijk} J_{jk} \epsilon_{ilr} J^{lr} |k;\sigma,s,n\rangle \\ &= -\sqrt{m/p^0}\, U[L(p)] m^2 \boldsymbol{J}^2 |k;\sigma,s,n\rangle \\ &= -\sqrt{m/p^0}\, U[L(p)] m^2 s\,(s+1)\,|k;\sigma,s,n\rangle \\ &= -m^2 s\,(s+1)\,|p;\sigma,s,n\rangle. \end{aligned} \tag{2.3.37}$$

The Poincaré invariant labels of the states $|p;\sigma,s,n\rangle$ as measured by the Casimir operators may thus be summarised by

$$C_1|p;\sigma,s,n\rangle = m^2|p;\sigma,s,n\rangle \quad \text{and} \quad C_2|p;\sigma,s,n\rangle = -m^2 s\,(s+1)\,|p;\sigma,s,n\rangle.$$

We thus have a frame-independent labelling of particle states in terms of two numbers: m^2 and $m^2 s\,(s+1)$, measured respectively by C_1 and C_2, the Casimir operators of the Poincaré group. It is worth noting that, although the eigenvalues of the operators $-C_2/m^2$ and $\boldsymbol{J}^2$ coincide in the rest frame, it would be false to draw the conclusion that spin is a frame-independent concept. The spin operator, $\boldsymbol{J}^2$, is not invariant under the unitary representations of the restricted Poincaré group. This is easy to see by looking at the commutation relations between $\boldsymbol{J}^2$ and generators of Lorentz boost. Ryder [82, p. 56] shows by explicit evaluation that $\left[\boldsymbol{J}^2, K_i\right] \neq 0$, for $i \in \{x,y,z\}$, and refers to [85–87] for a relativistic treatment of spin.

The spin projection index σ is measured in general by $U[L(p)]J_zU^{-1}[L(p)]$. This is easily demonstrated:

$$\begin{aligned} U[L(p)]J_zU^{-1}[L(p)]|p;\sigma,s,n\rangle &= U[L(p)]J_zU^{-1}[L(p)]\sqrt{m/p^0}U[L(p)]|k;\sigma,s,n\rangle \\ &= \sqrt{m/p^0}\,U[L(p)]J_z|k;\sigma,s,n\rangle \\ &= \sigma|p;\sigma,s,n\rangle. \end{aligned}$$

Recalling that J_z fails to commute with K_x and K_y, it follows that the spin projection of a given state will change under the action of any non-trivial Lorentz boost that is non-colinear with the original polarization of the state.

Massless particle

The case of a massless particle of positive energy is somewhat peripheral in the context of the present work. We therefore refrain from a detailed discussion and instead refer to [40, p. 72] where the following results are given. The states at standard momentum $k = (\kappa, 0, 0, \kappa)$ are diagonalised such that

$$A|k;\sigma\rangle = B|k;\sigma\rangle = 0, \tag{2.3.38}$$

$$J_z|k;\sigma\rangle = \sigma|k;\sigma\rangle, \tag{2.3.39}$$

where the operators A and B are as defined in (2.1.21). The transformation property under the action of unitary representations of $\mathcal{P}_+^\uparrow$ reads

$$U[\Lambda, a]|p;\sigma\rangle = \sqrt{(\Lambda p)^0/p^0}\,\mathrm{e}^{\Lambda p\cdot a}\,|\Lambda p;\sigma\rangle\,\mathrm{e}^{i\sigma\theta(\Lambda,p)}, \tag{2.3.40}$$

where $\theta(\Lambda, p)$ is the angle associated with the rotation generator J_z. We will briefly return to massless particles of positive energy in Ch. 6 where two fields will be explicitly constructed for the description of particles and antiparticles of helicity $\sigma = \pm 1/2$. For the remainder of the present chapter we will focus on the case of a massive particle of positive energy.

This concludes our exposition of the transformation properties of the single particle states under the action of the continuous symmetries of the restricted Poincaré group.

2.3.2 Discrete symmetries

Having in the previous section obtained a representation of the restricted Poincaré group by means of the method of induced representations, we now seek to derive representations of the discrete symmetries. Although it is experimentally well established that the discrete symmetries of space-inversion and time-reversal defined in (2.1.5) are violated in weak interactions [88–93] they remain good approximate symmetries [40, p. 75].

We choose here to begin with CPT, a symmetry inherent to all local Lorentz covariant quantum field theories by the CPT theorem [94–96]. The reason for this slightly unconventional approach is twofold. Unlike the symmetries of space-inversion and time-reversal, the symmetry of charge-conjugation, denoted C, does not arise in the study of the transformation properties of Minkowski space. This is, however, by no means an insuperable difficulty thanks to the known action of CPT [40, p. 103]. Once CPT, P, and T have been defined, the charge-conjugation operator is given by $(CPT)T^{-1}P^{-1}$. A further advantage of the approach here taken is that it will allow for the degeneracy index n to be identified before we explore the transformation properties of the single particle states under P and T.

CPT

Weinberg motivates the existence of antiparticles in [97, p. 61–63] by the demand that causality be preserved in the unification of special relativity and quantum mechanics. In [40, p. 244–245] he makes the following remark:

> Not only is it necessary that every particle have an antiparticle (which may for a purely neutral particle be itself); there is a precise relation between the properties of particles and antiparticles, that can be summarised in the statement that *for an appropriate choice of inversion phases, the product CPT of all the inversions is conserved.*

This implies that there exists an operator CPT whereby the above defined state vectors have the transformation property [40, p. 103]

$$CPT|p;\sigma,s,n\rangle = (-)^{s-\sigma}\,|p;-\sigma,s,n^c\rangle, \tag{2.3.41}$$

where the superscript "c" on the right hand side indicates that the state is that of an antiparticle [40, p. 104]. For the sake of notational simplicity we will write from hence forth $|p;\sigma,s,n\rangle \equiv |p;\sigma,s\rangle$ and $|p;\sigma,s,n^c\rangle \equiv |p;\sigma,s\rangle^c$. Of course there may be yet other degeneracies of the states that are broken only when the corresponding symmetries are considered. We shall not concern ourselves with this question here.

Given that the demand of CPT conservation applies to all particles (including antiparticles), the above relation must apply equally to antiparticle states. In summary, we thus have

$$CPT|p;\sigma,s\rangle = (-)^{s-\sigma}\,|p;-\sigma,s\rangle^c, \tag{2.3.42}$$

$$CPT|p;\sigma,s\rangle^c = (-)^{s-\sigma}\,|p;-\sigma,s\rangle. \tag{2.3.43}$$

Weinberg emphasises that no phases or matrices are permitted in the action of CPT on the state space [40, p. 104]. This has the immediate consequence that $(CPT)^2$ returns a state to itself with sign $+1$ for a state of integer spin and -1 for a state of half-integer spin; that is,

$$(CPT)^2\,|p;\sigma,s\rangle = (-)^{2s}\,|p;\sigma,s\rangle, \tag{2.3.44}$$

and likewise for an antiparticle state $|p;\sigma,s\rangle^c$.

As with P and T, CPT must be either unitary and linear or antiunitarity and antilinear. Authoritative works on the subject [40, 98, 99] take CPT (or some equivalent succession of discrete symmetries) to be antiunitarity and antilinear. We here show that this follows unequivocally if we postulate

$$(CPT)\, U[\Lambda, a]\, (CPT)^{-1} = \vartheta_4 U[\mathscr{Q}\Lambda\mathscr{Q}^{-1}, \mathscr{Q}a], \tag{2.3.45}$$

as the behaviour of the unitary representations of the restricted Poincaré group under the action of CPT by conjugation. Here $\mathscr{Q}$ is some yet to be determined 4×4 matrix that maps Minkowski space into itself; $\vartheta_4 = \vartheta_4(\mathscr{Q}, \Lambda, a, \mathscr{Q}^{-1})$ is a phase that shall be constrained in due course.

We begin by noting the following: it is implicit in the above identification of the degeneracy index n, with the notational distinction between particles and antiparticles, that particle state vectors transform identically as compared to antiparticle state vectors under the action of the unitary representations of $\mathcal{P}_+^\uparrow$. This is in agreement with a remark by Weinberg in [64]: "If an antiparticle exists then its states will transform like those of the corresponding particle." It thus follows from (2.3.2), that

$$P^\mu|p;\sigma,s\rangle \;= p^\mu|p;\sigma,s\rangle, \tag{2.3.46}$$

$$P^\mu|p;\sigma,s\rangle^c = p^\mu|p;\sigma,s\rangle^c. \tag{2.3.47}$$

Applying CPT to (2.3.46) and using (2.3.42), we obtain

$$CPTP^\mu|p;\sigma,s\rangle = p^\mu CPT|p;\sigma,s\rangle = p^\mu\,(-)^{s-\sigma}\,|p;-\sigma,s\rangle^c = P^\mu\,(-)^{s-\sigma}\,|p;-\sigma,s\rangle^c.$$

Equivalently, we have

$$CPTP^\mu\,(CPT)^{-1}\,CPT|p;\sigma,s\rangle = CPTP^\mu\,(CPT)^{-1}\,(-)^{s-\sigma}\,|p;-\sigma,s\rangle^c.$$

Together, these imply that the energy-momentum operator P^μ commutes with CPT:

$$(CPT)\,P^\mu\,(CPT)^{-1} = P^\mu. \tag{2.3.48}$$

By the same procedure that led to (2.2.31) and (2.2.32) for parity, again taking the phase in (2.3.45) to be a continuous function of the continuous symmetries Λ and a [49, p. 169], we obtain the following transformation properties for the ten generators of the Lie algebra of the restricted Poincaré group under the conjugate action of CPT:

$$(CPT)\,iJ^{\mu\nu}\,(CPT)^{-1} = i\mathscr{Q}_\alpha{}^\mu\mathscr{Q}_\beta{}^\nu J^{\alpha\beta}, \tag{2.3.49}$$

$$(CPT)\,iP^\mu\,(CPT)^{-1} = i\mathscr{Q}_\alpha{}^\mu P^\alpha. \tag{2.3.50}$$

Taking (2.3.48) together with (2.3.50), one has

$$iP^\mu = i\mathscr{Q}_\alpha{}^\mu P^\alpha \quad \Longrightarrow \quad \mathscr{Q}_\nu{}^\mu = +\delta_\nu{}^\mu, \quad \text{for } CPT \text{ unitary and linear,} \tag{2.3.51}$$

$$-iP^\mu = i\mathscr{Q}_\alpha{}^\mu P^\alpha \quad \Longrightarrow \quad \mathscr{Q}_\nu{}^\mu = -\delta_\nu{}^\mu, \quad \text{for } CPT \text{ antiunitary and antilinear.} \tag{2.3.52}$$

Accordingly, from (2.3.49), we obtain

$$(CPT)\, J^{\mu\nu}\, (CPT)^{-1} = +J^{\mu\nu}, \quad \text{for } CPT \text{ unitary and linear,} \tag{2.3.53}$$

$$(CPT)\, J^{\mu\nu}\, (CPT)^{-1} = -J^{\mu\nu}, \quad \text{for } CPT \text{ antiunitary and antilinear.} \tag{2.3.54}$$

Again, because particle and antiparticle states transform in the same manner under the unitary representations of the restricted Poincaré group, we have

$$J_z|k;\sigma,s\rangle = \sigma|k;\sigma,s\rangle, \tag{2.3.55}$$

$$J_z|k;\sigma,s\rangle^c = \sigma|k;\sigma,s\rangle^c. \tag{2.3.56}$$

Applying CPT to (2.3.55), we find

$$CPTJ_z|k;\sigma,s\rangle = \sigma CPT|k;\sigma,s\rangle = \sigma\,(-)^{s-\sigma}\,|k;-\sigma,s\rangle^c = -J_z\,(-)^{s-\sigma}\,|k;-\sigma,s\rangle^c.$$

Equivalently, we have

$$CPTJ_z|k;\sigma,s\rangle = CPTJ_z\,(CPT)^{-1}\,(-)^{s-\sigma}\,|k;-\sigma,s\rangle^c$$

Together, these imply that J_z anticommutes with CPT:

$$(CPT)\, J_z\, (CPT)^{-1} = -J_z. \tag{2.3.57}$$

Recalling, from the identification given in (2.2.20), that $J^{21} \equiv J_z$ and comparing (2.3.57) with (2.3.54), it is clear that CPT must be antiunitary and antilinear. Consequently from (2.3.52), we have

$$\mathscr{Q} = -(\delta_\nu{}^\mu) = \mathscr{P}\mathscr{T}. \tag{2.3.58}$$

Furthermore, by (2.3.54), $J^{\mu\nu}$ anticommutes with CPT:

$$(CPT)\, J^{\mu\nu}\, (CPT)^{-1} = -J^{\mu\nu}. \tag{2.3.59}$$

With the CPT operator thus defined, we may now return to (2.3.45) so as to constrain the phase ϑ_4. To this end we first rewrite (2.3.45) as

$$(CPT)\, U[\Lambda,a] = \vartheta_4 U[\Lambda,-a]\,(CPT)\,. \tag{2.3.60}$$

We now apply both sides to the one-particle state $|p;\sigma,s\rangle$.

Beginning with the LHS we use the transformation property under $U[\Lambda, a]$, as given in (2.3.34), followed by (2.3.42), along with the antiunitarity antilinear property of CPT, to obtain

$$(CPT)\,U[\Lambda, a]\,|p;\sigma, s\rangle = (CPT)\,\mathrm{e}^{i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}|\Lambda p;\sigma', s\rangle D^{(s)}_{\sigma'\sigma}[W(\Lambda, p)]$$
$$= \mathrm{e}^{-i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}|\Lambda p;-\sigma', s\rangle^c D^{*(s)}_{\sigma'\sigma}[W(\Lambda, p)](-)^{s-\sigma'}.$$

Applying the right hand side (RHS) of (2.3.60) to the one-particle state yields

$$\vartheta_4 U[\Lambda, -a]\,(CPT)\,|p;\sigma, s\rangle = \vartheta_4 U[\Lambda, -a](-)^{s-\sigma}|p;-\sigma, s\rangle^c$$
$$= \vartheta_4(-)^{s-\sigma}\,\mathrm{e}^{-i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}|\Lambda p;-\sigma', s\rangle^c D^{(s)}_{-\sigma'-\sigma}[W(\Lambda, p)].$$

Equating and cancelling common factors we thus have

$$\sum_{\sigma'}|\Lambda p;-\sigma', s\rangle^c D^{*(s)}_{\sigma'\sigma}[W(\Lambda, p)](-)^{\sigma-\sigma'} = \vartheta_4\sum_{\sigma'}|\Lambda p;-\sigma', s\rangle^c D^{(s)}_{-\sigma'-\sigma}[W(\Lambda, p)].$$

As noted by Weinberg in [40, p. 234], the generators (2.3.30) satisfy the relation

$$(-)^{\sigma-\sigma'}J^{*(s)}_{\sigma'\sigma} = -J^{(s)}_{-\sigma'-\sigma}. \tag{2.3.61}$$

Considering the factor of i in the expansion (2.3.33), it thus follows that the matrices $D[W(\Lambda, a)]$ obey

$$(-)^{\sigma-\sigma'}D^{*(s)}_{\sigma'\sigma}[W(\Lambda, a)] = D^{(s)}_{-\sigma'-\sigma}[W(\Lambda, a)]. \tag{2.3.62}$$

We conclude that $\vartheta_4 = 1$; accordingly, the action of CPT on $U[\Lambda, a]$ by conjugation becomes

$$(CPT)\,U[\Lambda, a]\,(CPT)^{-1} = U[\Lambda, -a]. \tag{2.3.63}$$

Having uniquely determined the properties of CPT, we now deduce the action of P and T from their known properties on Minkowski space. Thereafter we shall consider the resultant properties of charge-conjugation.

Space-inversion

To find the action of parity on the state space, we first recall from (2.2.41) that parity commutes with the angular momentum generators and with the energy operator. Consequently,

$$\begin{aligned} HP|k;\sigma,s\rangle &= PH|k;\sigma,s\rangle = m\,P|k;\sigma,s\rangle, \\ J_zP|k;\sigma,s\rangle &= PJ_z|k;\sigma,s\rangle = \sigma\,P|k;\sigma,s\rangle, \\ \boldsymbol{J}^2P|k;\sigma,s\rangle &= P\boldsymbol{J}^2|k;\sigma,s\rangle = s\,(s+1)\;P|k;\sigma,s\rangle, \end{aligned}$$

indicating that $P|k;\sigma,s\rangle$, like $|k;\sigma,s\rangle$, is an eigenstate of H, J_z, and J^2. It must therefore be equal to $|k;\sigma,s\rangle$ up to a phase which may or may not depend on the quantum numbers of the state. We thus have

$$P|k;\sigma,s\rangle = \xi_{k,\sigma,s}|k;\sigma,s\rangle. \tag{2.3.64}$$

The phase, $\xi_{k,\sigma,s}$, is known as the intrinsic parity [100, p. 572]. To establish the dependence of the phase on σ, recall from (2.3.27) and (2.3.28) that

$$(J_x \pm iJ_y)\,|k;\sigma,s\rangle = \sqrt{(s\mp\sigma)\,(s\pm\sigma+1)}\;|k;\sigma\pm 1,s\rangle. \tag{2.3.65}$$

Applying P on both sides, we obtain

$$P\,(J_x \pm iJ_y)\,P^{-1}P|k;\sigma,s\rangle = \sqrt{(s\mp\sigma)\,(s\pm\sigma+1)}\;P|k;\sigma\pm 1,s\rangle. \tag{2.3.66}$$

Given that P is a unitary linear operator that commutes with the rotation generators, this implies that

$$\xi_{k,\sigma,s} = \xi_{k,\sigma\pm 1,s}. \tag{2.3.67}$$

The intrinsic parity must therefore be independent of spin projection σ; at most it may depend on the spin and the rest mass of the state under consideration. We henceforth omit the k index because this will be assumed to be of a fixed value for any given theory under consideration. The states at rest thus have the transformation property

$$P|k;\sigma,s\rangle = \xi_s\,|k;\sigma,s\rangle. \tag{2.3.68}$$

As to the action of parity on a state at momentum p, we find from (2.3.36) and (2.2.28), and the identity $\mathscr{P}L(p)\mathscr{P}^{-1} = L(\mathscr{P}p)$, that

$$\begin{aligned} P|p;\sigma,s\rangle &= \sqrt{m/p^0}\,P\,U[L(p)]P^{-1}P\,|k;\sigma,s\rangle \\ &= \vartheta_5\xi_s\sqrt{m/p^0}\,U[\mathscr{P}L(p)\mathscr{P}^{-1}]\,|k;\sigma,s\rangle \\ &= \vartheta_5\xi_s\sqrt{m/p^0}\,U[L(\mathscr{P}p)]\,|k;\sigma,s\rangle \\ &= \vartheta_5\xi_s\,|\mathscr{P}p;\sigma,s\rangle. \end{aligned} \tag{2.3.69}$$

We now take a closer look at the phase ϑ_5. Here it is essential to explicitly display the alleged functional dependence. We thus rewrite (2.2.28) as

$$PU[\Lambda,a] = \vartheta_5(\mathscr{P},\Lambda,a,\mathscr{P})U[\mathscr{P}\Lambda\mathscr{P}^{-1},\mathscr{P}a]P, \tag{2.3.70}$$

where we have used that $\mathscr{P}$ is involutory. Applying the LHS of (2.3.70) to the one particle state $|p;\sigma,s\rangle$, we obtain

$$\begin{aligned}PU[\Lambda,a]|p;\sigma,s\rangle &= P\,\mathrm{e}^{i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}|\Lambda p;\sigma',s\rangle D^{(s)}_{\sigma'\sigma}[W(\Lambda,p)]\\ &= \vartheta_5(\mathscr{P},L(\Lambda p),a,\mathscr{P})\xi_s\,\mathrm{e}^{i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}|\mathscr{P}\Lambda p;\sigma',s\rangle D^{(s)}_{\sigma'\sigma}[W(\Lambda,p)].\end{aligned}$$

Similarly for the RHS of (2.3.70) we find

$$\begin{aligned}&\vartheta_5(\mathscr{P},\Lambda,a,\mathscr{P})U[\mathscr{P}\Lambda\mathscr{P}^{-1},\mathscr{P}a]P|p;\sigma,s\rangle\\ &\quad= \vartheta_5(\mathscr{P},\Lambda,a,\mathscr{P})\vartheta_5(\mathscr{P},L(p),0,\mathscr{P})\xi_s U[\mathscr{P}\Lambda\mathscr{P}^{-1},\mathscr{P}a]|\mathscr{P}p;\sigma,s\rangle\\ &\quad= \vartheta_5(\mathscr{P},\Lambda,a,\mathscr{P})\vartheta_5(\mathscr{P},L(p),0,\mathscr{P})\xi_s\,\mathrm{e}^{i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\\ &\quad\times\sum_{\sigma'}|\mathscr{P}\Lambda p;\sigma',s\rangle D^{(s)}_{\sigma'\sigma}[W(\mathscr{P}\Lambda\mathscr{P}^{-1},\mathscr{P}p)].\end{aligned}$$

But

$$W(\mathscr{P}\Lambda\mathscr{P}^{-1},\mathscr{P}p) = \mathscr{P}W(\Lambda,p)\mathscr{P}^{-1} = W(\Lambda,p). \tag{2.3.71}$$

We thus obtain the following relationship between the phases

$$\vartheta_5(\mathscr{P},L(\Lambda p),a,\mathscr{P}) = \vartheta_5(\mathscr{P},\Lambda,a,\mathscr{P})\vartheta_5(\mathscr{P},L(p),0,\mathscr{P}). \tag{2.3.72}$$

We shall not explore this matter any further in the present section. For the remainder of the work, we make the choice $\vartheta_5 = 1$. This is consistent with a theorem by Streater and Wightman [98, p. 127]. The composition rule then becomes

$$PU[\Lambda,a]P^{-1} = U[\mathscr{P}\Lambda\mathscr{P}^{-1},\mathscr{P}a]. \tag{2.3.73}$$

Given that in deriving (2.3.69) we used nothing but the properties of the particle states under the symmetries of the Poincaré group and the previously derived properties of parity on Minkowski space, it follows that the transformation properties of the antiparticle states must be of the same form as those derived above. In summary, we thus have

$$P|p;\sigma,s\rangle \;= \xi_s|\mathscr{P}p;\sigma,s\rangle, \tag{2.3.74}$$

$$P|p;\sigma,s\rangle^c = \xi^c_s|\mathscr{P}p;\sigma,s\rangle^c. \tag{2.3.75}$$

The phases ξ_s and ξ^c_s can be independently chosen because there remains the possibility of a phase freedom in the composition rule for the operators of space-inversion and CPT.

With the transformation properties of the state space under the action of the unitary space-inversion operator thus defined, we may check whether the operator $P \equiv U[\mathscr{P}]$ satisfies the composition rule for the underlying involutory Minkowski space symmetry $\mathscr{P}$. Taking (2.3.74) and applying parity for a second time, we obtain

$$U[\mathscr{P}]U[\mathscr{P}]|p;\sigma,s\rangle = U[\mathscr{P}]\xi_s|\mathscr{P}p;\sigma,s\rangle = \xi_s\xi_s|p;\sigma,s\rangle = \xi_s\xi_s U[\mathscr{P}\mathscr{P}]|p;\sigma,s\rangle.$$

We thus have the composition rule

$$U[\mathscr{P}]U[\mathscr{P}] = \xi_s\xi_s U[\mathscr{P}\mathscr{P}], \tag{2.3.76}$$

consistent with that of the Minkowski space operators. It is thus true to say that the unitary operator $P \equiv U[\mathscr{P}]$, defined by (2.3.74), induces a projective representation of space-inversion on the state space. If by whatever means the intrinsic parity, ξ_s, for a given particle were found to be real, the representation of space-inversion on the state space of that particle would be non-projective.

Time-reversal

In order to discern the action of time-reversal on the state space we begin by looking at the particle states at rest defined in (2.3.22)–(2.3.24). Recalling that $THT^{-1} = H$, $T\boldsymbol{J}T^{-1} = -\boldsymbol{J}$, and $T\boldsymbol{J}^2T^{-1} = \boldsymbol{J}^2$, we have

$$\begin{aligned} HT|k;\sigma,s\rangle &= TH|\sigma,s\rangle = mT|k;\sigma,s\rangle, \\ J_zT|k;\sigma,s\rangle &= -TJ_z|\sigma,s\rangle = -\sigma\, T|k;\sigma,s\rangle, \\ \boldsymbol{J}^2T|k;\sigma,s\rangle &= T\boldsymbol{J}^2|\sigma,s\rangle = s\,(s+1)\,T|k;\sigma,s\rangle. \end{aligned}$$

Hence $T|k;\sigma,s\rangle$ is an eigenstate of H, J_z, and $\boldsymbol{J}^2$, albeit with the spin projection index reversed in sign. The time-reversed state must thus be proportional to the original state up to a phase and with the index σ reversed in sign:

$$T|k;\sigma,s\rangle = \zeta_{k,\sigma,s}|k;-\sigma,s\rangle. \tag{2.3.77}$$

As yet, we cannot exclude the possibility that the time-reversal phase may depend on the indices of the state under consideration. We can, however, determine its precise dependence on the spin projection index σ. Following a similar approach as was applied in the previous section, we invoke the raising and lowering operators and recall, from (2.3.65), the following relation:

$$(J_x \pm iJ_y)\,|k;\sigma,s\rangle = \sqrt{(s\mp\sigma)\,(s\pm\sigma+1)}\;|k;\sigma\pm1,s\rangle. \tag{2.3.78}$$

Switching $\pm$ to $\mp$ and exchanging σ with $-\sigma$ on both sides yields

$$(J_x \mp iJ_y)\,|k;-\sigma,s\rangle = \sqrt{(s\mp\sigma)\,(s\pm\sigma+1)}\;|k;-\sigma\mp1,s\rangle. \tag{2.3.79}$$

Interchanging the RHS with the LHS in (2.3.78) and applying T on both sides, we obtain from (2.2.41) and the antilinearity of the time-reversal operator on the state space

$$\begin{aligned}\sqrt{(s\mp\sigma)\,(s\pm\sigma+1)}\;T|k;\sigma\pm 1,s\rangle &= T\,(J_x\pm iJ_y)\,|k;\sigma,s\rangle\\ &= (-J_x\pm iJ_y)\,T|k;\sigma,s\rangle\\ &= \zeta_{k,\sigma,s}\,(-J_x\pm iJ_y)\,|k;-\sigma,s\rangle\\ &= -\zeta_{k,\sigma,s}\sqrt{(s\mp\sigma)\,(s\pm\sigma+1)}\;|k;-\sigma\mp 1,s\rangle,\end{aligned}$$

where, for the last step, we made use of (2.3.79). Hence, except for the trivial case in which $s=0$, we must have

$$-\zeta_{k,\sigma,s}|k;-\sigma\mp 1,s\rangle = T|k;\sigma\pm 1,s\rangle = \zeta_{k,\sigma\pm 1,s}|k;-\sigma\mp 1,s\rangle.$$

Therefore, the time-reversal phase must satisfy the relation

$$-\zeta_{k,\sigma,s} = \zeta_{k,\sigma\pm 1,s}. \tag{2.3.80}$$

Weinberg provides [40, p. 78] a solution of the form $\zeta_{k,\sigma,s} = \zeta_{k,s}(-)^{s-\sigma}$, albeit without the explicit indices s and k on the phase. It is easy to verify this solution by direct substitution:

$$-\zeta_{k,\sigma,s} = -\zeta_{k,s}(-)^{s-\sigma} = \zeta_{k,s}(-)^{s-\sigma}(-)^{\mp 1} = \zeta_{k,s}(-)^{s-(\sigma\pm 1)} = \zeta_{k,\sigma\pm 1,s}.$$

Therefore (2.3.80) is satisfied, as was to be shown. The transformation of the rest states given in (2.3.77) thus becomes

$$T|k;\sigma,s\rangle = \zeta_{k,s}(-)^{s-\sigma}|k;-\sigma,s\rangle. \tag{2.3.81}$$

The explicit rest-mass dependence of the phase will henceforth be omitted for the sake of notational convenience. As argued above in the case of the rest mass dependence of the intrinsic parity, the simplification is of no consequence because the theories to be constructed in the present work will each involve a single fixed value for the rest mass.

To find the transformation properties of the states at momentum p, we apply T to (2.3.36) and use the composition rule given in (2.2.29) as well as the identity $\mathscr{T}L(p)\mathscr{T}^{-1} = L(\mathscr{P}p)$ derived in App. B.3. We thereby obtain

$$\begin{aligned}T|p;\sigma,s\rangle &= \sqrt{m/p^0}\,TU[L(p)]T^{-1}T|k,\sigma,s\rangle\\ &= \sqrt{m/p^0}\,U[\mathscr{T}L(p)\mathscr{T}^{-1}]\vartheta_6\zeta_{k,s}(-)^{s-\sigma}|k,-\sigma,s\rangle\\ &= \vartheta_6\zeta_s\,(-)^{s-\sigma}\sqrt{m/p^0}\,U[L(\mathscr{P}p)]|k,-\sigma,s\rangle\\ &= \vartheta_6\zeta_s\,(-)^{s-\sigma}|\mathscr{P}p;-\sigma,s\rangle.\end{aligned} \tag{2.3.82}$$

With respect to the phase ϑ_6 we note that it is straight forward, by means akin to those applied in the previous case of space-inversion, to show that the following relation must hold

$$\vartheta_6(\mathscr{T}, L(\Lambda p), a, \mathscr{T}) = \vartheta_6(\mathscr{T}, \Lambda, a, \mathscr{T})\vartheta_6(\mathscr{T}, L(p), 0, \mathscr{T}). \tag{2.3.83}$$

A detailed analysis of phases such as the present ϑ_6 and the former ϑ_5, that arise in representations of the full Poincaré group, is given by Sternberg [9, Sec. 3.10]. In accordance with a theorem by Streater and Wightman [98, p. 127], we take $\vartheta_6 = 1$. The composition rule (2.2.29) thus becomes

$$TU[\Lambda, a]T^{-1} = U[\mathscr{T}\Lambda\mathscr{T}^{-1}, \mathscr{T}a]. \tag{2.3.84}$$

With this simplification, we summarise the transformation property of the one particle states as follows:

$$T|p;\sigma,s\rangle \; = \zeta_s(-)^{s-\sigma}|\mathscr{P}p;-\sigma,s\rangle, \tag{2.3.85}$$

$$T|p;\sigma,s\rangle^c = \zeta_s^c(-)^{s-\sigma}|\mathscr{P}p;-\sigma,s\rangle^c, \tag{2.3.86}$$

where the transformation of the antiparticle state may be derived by the exact same technique as demonstrated above for the particle state. The phases ζ_s and ζ_s^c can be chosen independently of one another provided there is a freedom in the choice of phase in the composition rule of T with CPT.

Before we move on to explore the symmetry of charge-conjugation, let us check whether the operator $T \equiv U[\mathscr{T}]$, defined by the above derived transformation properties of the one particle states, satisfies the composition rule of the underlying Minkowski space transformation. Applying time-reversal to (2.3.85), we obtain

$$\begin{aligned} U[\mathscr{T}]U[\mathscr{T}]|p;\sigma,s\rangle &= U[\mathscr{T}]\zeta_s(-)^{s-\sigma}|\mathscr{P}p;-\sigma,s\rangle \\ &= \zeta_s^*(-)^{s-\sigma}U[\mathscr{T}]|\mathscr{P}p;-\sigma,s\rangle \\ &= \zeta_s^*(-)^{s-\sigma}\zeta_s(-)^{s+\sigma}|p;\sigma,s\rangle \\ &= (-)^{2s}U[\mathscr{T}\mathscr{T}]|p;\sigma,s\rangle, \end{aligned}$$

thus implying the composition rule

$$U[\mathscr{T}]U[\mathscr{T}] = (-)^{2s}U[\mathscr{T}\mathscr{T}]. \tag{2.3.87}$$

For a particle of half-integer spin, we thus have a projective representation of time-reversal on the state space; for a particle of integer spin, the representation is non-projective.

Charge-conjugation

To the extent that the action of the operator CPT, defined above in Sec. 2.3.2, is interpreted as that of time-reversal followed by space-inversion followed by the here to be determined symmetry of charge-conjugation, we may uniquely express the latter as

$$C \equiv (CPT)T^{-1}P^{-1}. \tag{2.3.88}$$

With this identification it immediately follows that C must be unitary and linear because it is given by the composition of two antiunitary antilinear operators and one unitary linear operator. Furthermore, the behaviour of $U[\Lambda, a]$ under conjugation by C can be deduced by noting, from (2.2.28) and (2.2.29), that

$$(CPT)U[\Lambda, a](CPT)^{-1} = CPTU[\Lambda, a]T^{-1}P^{-1}C^{-1} = \vartheta_5\vartheta_6 CU[\Lambda, -a]C^{-1}.$$

Recalling the choice $\vartheta_5 = \vartheta_6 = 1$ and invoking (2.3.63), we obtain

$$CU[\Lambda, a]C^{-1} = U[\Lambda, a]. \tag{2.3.89}$$

In order to determine the action of charge-conjugation on the one particle states, we first require the transformation of the latter under P^{-1} and T^{-1}. As can be easily deduced respectively from (2.3.85) and (2.3.74), we have

$$P^{-1}|p; \sigma, s\rangle = \xi_s^*|\mathscr{P}p; \sigma, s\rangle, \tag{2.3.90}$$

$$T^{-1}|p; \sigma, s\rangle = \zeta_s\,(-)^{-s-\sigma}\,|\mathscr{P}p; -\sigma, s\rangle. \tag{2.3.91}$$

The action of $(CPT)T^{-1}P^{-1}$ on the one particle states thus reads

$$\begin{aligned}(CPT)\,T^{-1}P^{-1}|p; \sigma, s\rangle &= (CPT)\,\xi_s\zeta_s\,(-)^{-s-\sigma}\,|p; -\sigma, s\rangle\\ &= \xi_s^*\zeta_s^*\,(-)^{-s-\sigma}\,(CPT)\,|p; -\sigma, s\rangle\\ &= \xi_s^*\zeta_s^*\,(-)^{-s-\sigma}\,(-)^{s+\sigma}\,|p; \sigma, s\rangle^c.\end{aligned} \tag{2.3.92}$$

Up to a phase factor $\xi_s^*\zeta_s^*$, the action of charge-conjugation on the state space is thus simply the interchange of particles states and antiparticle states. Repeating the above derivation for the state $|p; \sigma, s\rangle^c$ yields the relevant counterpart of (2.3.92). The results are summarised by

$$C|p; \sigma, s\rangle \;= \xi_s^*\zeta_s^*|p; \sigma, s\rangle^c, \tag{2.3.93}$$

$$C|p; \sigma, s\rangle^c = \xi_s^{c*}\zeta_s^{c*}|p; \sigma, s\rangle. \tag{2.3.94}$$

Looking at (2.3.93) and (2.3.94), we see that the charge-conjugation phase factor is related to the product of the intrinsic parity, ξ_s, and the time-reversal phase, ζ_s, by complex conjugation. The charge-conjugation phase factor therefore does not constitute an independent degree of freedom.

CP, CT, and PT

The succession of discrete symmetries CP is given by

$$\begin{aligned}CP\,|p; \sigma, s\rangle &= C\xi_s|\mathscr{P}p; \sigma, s\rangle\\ &= \xi_s\xi_s^*\zeta_s^*|\mathscr{P}p; \sigma, s\rangle^c\\ &= \zeta_s^*|\mathscr{P}p; \sigma, s\rangle^c.\end{aligned} \tag{2.3.95}$$

The succession of discrete symmetries CT is given by

$$\begin{aligned} CT\,|p;\sigma,s\rangle &= C\zeta_s\,(-)^{s-\sigma}\,|\mathscr{P}p;-\sigma,s\rangle \\ &= \zeta_s\,(-)^{s-\sigma}\,\xi_s^*\zeta_s^*|\mathscr{P}p;-\sigma,s\rangle^c \\ &= \xi_s^*\,(-)^{s-\sigma}\,|\mathscr{P}p;-\sigma,s\rangle^c. \end{aligned} \tag{2.3.96}$$

The succession of discrete symmetries PT is given by

$$\begin{aligned} PT\,|p;\sigma,s\rangle &= P\zeta_s\,(-)^{s-\sigma}\,|\mathscr{P}p;-\sigma,s\rangle \\ &= \zeta_s\,(-)^{s-\sigma}\,\xi_s|p;-\sigma,s\rangle \\ &= \xi_s\zeta_s\,(-)^{s-\sigma}\,|p;-\sigma,s\rangle. \end{aligned} \tag{2.3.97}$$

2.3.3 The vacuum and open questions on phases

The state space vector containing no particles is called the vacuum state, or simply the vacuum, and is denoted by $|\ \rangle$. It is unique up to a constant phase [98, p. 97] and is of unit norm:

$$\langle\ |\ \rangle = 1. \tag{2.3.98}$$

Furthermore, it is chosen to be invariant under the action of the unitary representations of the restricted Poincaré group:

$$U[\Lambda,a]|\ \rangle = |\ \rangle, \qquad \forall\,\{\Lambda,a\}\in\mathcal{P}_+^\uparrow. \tag{2.3.99}$$

For the discrete symmetries, the vacuum must be invariant up to a phase:

$$U[\mathscr{P},0]|\ \rangle = \vartheta_p|\ \rangle, \quad \text{with} \quad |\vartheta_p| = 1, \tag{2.3.100}$$

$$U[\mathscr{T},0]|\ \rangle = \vartheta_t|\ \rangle, \quad \text{with} \quad |\vartheta_t| = 1, \tag{2.3.101}$$

$$C|\ \rangle = \vartheta_c|\ \rangle, \quad \text{with} \quad |\vartheta_c| = 1. \tag{2.3.102}$$

Explicit mention is made of this phase freedom in the literature. Lee and Wick, in their famous work of 1966, point out their assumption about the invariance of the vacuum state [99]. Streater and Wightman remark that it is standard convention to choose the phase factors in such a fashion that the vacuum is invariant under the above discrete symmetries [98, p. 128]. The choice

$$\vartheta_p = \vartheta_t = \vartheta_c = 1, \tag{2.3.103}$$

is also implicit in the treatment of Weinberg [40, p. 177]. We shall adhere to this convention in what is to follow.

This concludes the discussion of the symmetries of the space of physical states. In the next section, we will introduce creation and annihilation operators which, thanks to the invariance of the vacuum as per (2.3.99) and (2.3.103), will directly inherit the transformation properties of the state vectors and allow us later to bring these transformations to bear in the construction of quantum fields.

2.4 Creation and annihilation operators

We now introduce two new operators on the state space: the creation operator, $a^\dagger(p;\sigma,s)$, and the annihilation operator, $a(p;\sigma,s)$, defined with respect to their action on the vacuum state by

$$a^\dagger(p;\sigma,s)|\ \rangle = |p;\sigma,s\rangle, \tag{2.4.1}$$

$$a(p;\sigma,s)|\ \rangle = 0. \tag{2.4.2}$$

When applied to a multi-particle state, $|q_1, q_2, \ldots, q_N\rangle$ containing N particles defined respectively by quantum numbers $q_i \equiv \{p_i;\sigma_i,s_i\}$, the creation operator $a^\dagger(q)$ adds a particle with quantum number q: $a^\dagger(q)|q_1,\ldots,q_N\rangle = |q,q_1,\ldots,q_N\rangle$. The annihilation operator $a(q)$ removes a particle with quantum number q. For consistency with the particle interpretation of the states $|p;\sigma,s\rangle$, and the known properties of bosons and fermions, we demand that the creation and annihilation operators satisfy the commutation relations

$$\Big[a(p;\sigma,s),\, a^\dagger(p';\sigma',s')\Big]_\pm = \delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'}\delta_{ss'}, \tag{2.4.3}$$

where '$+$' denotes anticommutation, for fermionic statistics, and '$-$' denotes commutation, for bosonic statistics. We require a second set of operators for the creation and annihilation of antiparticle states; these are denoted by $b^\dagger(p;\sigma,s) \equiv a^{c\dagger}(p;\sigma,s)$ and $b(p;\sigma,s) \equiv a^c(p;\sigma,s)$, respectively. Their commutation relations read

$$\Big[b(p;\sigma,s),\, b^\dagger(p';\sigma',s')\Big]_\pm = \delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'}\delta_{ss'}. \tag{2.4.4}$$

All other (anti-) commutators, between the above defined operators, vanish.

Weinberg makes the following remark on the significance of the use of creation and annihilation operators in [40, p. 169]:

> The great advantage of this formalism is that if we express the Hamiltonian as a sum of products of creation and annihilation operators, with suitable non-singular coefficients, then the S-matrix will automatically satisfy a crucial physical requirement, the cluster decomposition principle, which says in effect that distant experiments yield uncorrelated results.

We will provide an explicit derivation of two free Hamiltonians in terms of the here defined creation and annihilation operators: once in Ch. 3 and, for a different theory, in Ch. 4. Before this can be achieved, we must first complete the present derivation of quantum fields. The remainder of the present section will involve a deduction of the transformation properties of the creation and annihilation operators under the action of the ten continuous and three discrete symmetries of the state space.

In order to derive the transformation properties of the creation and annihilation operators under the unitary representations of the restricted Poincaré group, we begin by recalling the transformation property of the one particle states, given in (2.3.34), along with the definition of the particle creation operator, given in (2.4.1). We thus obtain

$$\begin{aligned} U[\Lambda, a]\,|p;\sigma,s\rangle &= \mathrm{e}^{i\Lambda p\cdot a}\left[(\Lambda p)^0/p^0\right]^{1/2}\sum_{\sigma'} D^{(s)}_{\sigma'\sigma}[W(\Lambda,p)]\,|\Lambda p;\sigma',s\rangle \\ &= \mathrm{e}^{i\Lambda p\cdot a}\left[(\Lambda p)^0/p^0\right]^{1/2}\sum_{\sigma'} D^{(s)}_{\sigma'\sigma}[W(\Lambda,p)]\,a^\dagger(\Lambda p;\sigma',s)|\ \ \rangle. \end{aligned}$$

Using once more the definition of the particle creation operator followed by the Poincaré invariance of the vacuum, we find

$$U[\Lambda, a]\,|p;\sigma,s\rangle = U[\Lambda,a]a^\dagger(p;\sigma,s)|\ \ \rangle = U[\Lambda,a]a^\dagger(p;\sigma,s)U^{-1}[\Lambda,a]|\ \ \rangle.$$

Combining the above two results, we obtain the following transformation property for the particle creation operator:

$$U[\Lambda,a]a^\dagger(p;\sigma,s)U^{-1}[\Lambda,a] = \mathrm{e}^{i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'} D^{(s)}_{\sigma'\sigma}[W(\Lambda,p)]\,a^\dagger(\Lambda p;\sigma',s). \tag{2.4.5}$$

Given that the matrices $D^{(s)}[W(\Lambda,p)]$ furnish a unitary non-projective representation of W, (2.4.5) may be rewritten as

$$U[\Lambda,a]a^\dagger(p;\sigma,s)U^{-1}[\Lambda,a] = \mathrm{e}^{i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'} D^{(s)*}_{\sigma\sigma'}[W^{-1}(\Lambda,p)]\,a^\dagger(\Lambda p;\sigma',s). \tag{2.4.6}$$

This allows us to immediately read off the transformation property of the annihilation operator by taking the Hermitian adjoint on both sides of (2.4.6). This gives

$$U[\Lambda,a]a(p;\sigma,s)U^{-1}[\Lambda,a] = \mathrm{e}^{-i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'} D^{(s)}_{\sigma\sigma'}[W^{-1}(\Lambda,p)]\,a(\Lambda p;\sigma',s). \tag{2.4.7}$$

We have postulated that particle and antiparticle states must transform identically under the action of unitary representations of $\mathcal{P}^\uparrow_+$. It thus follows that the transformation properties of the corresponding creation and annihilation operators, namely $b^\dagger(p;\sigma,s)$ and $b(p;\sigma,s)$, must be identical to those given in (2.4.6) and (2.4.7). This is not so in the case of the discrete symmetries. Using the assumption of an invariant vacuum, we may repeat the above for C, P, and T to obtain the associated transformation properties of the creation and annihilation operators. Recalling the transformation properties of the particle states, $|p;\sigma,s\rangle$, derived on the preceding pages, we find the particle creation operator, $a^\dagger(p;\sigma,s)$, has the following properties:

$$Ca^{\dagger}(p;\sigma,s)C^{-1} = \xi_s^{*}\zeta_s^{*}\, b^{\dagger}(p;\sigma,s), \tag{2.4.8}$$

$$Pa^{\dagger}(p;\sigma,s)P^{-1} = \xi_s\, a^{\dagger}(\mathscr{P}p;\sigma,s), \tag{2.4.9}$$

$$Ta^{\dagger}(p;\sigma,s)T^{-1} = \zeta_s\, (-)^{s-\sigma}\; a^{\dagger}(\mathscr{P}p;-\sigma,s). \tag{2.4.10}$$

For completeness we give the following transformation properties for the succession of discrete symmetries:

$$(CP)\, a^{\dagger}(p;\sigma,s)\,(CP)^{-1} = \zeta_s^{*}\, b^{\dagger}(\mathscr{P}p;\sigma,s), \tag{2.4.11}$$

$$(CT)\, a^{\dagger}(p;\sigma,s)\,(CT)^{-1} = \xi_s^{*}\,(-)^{s-\sigma}\; b^{\dagger}(\mathscr{P}p;-\sigma,s), \tag{2.4.12}$$

$$(PT)\, a^{\dagger}(p;\sigma,s)\,(PT)^{-1} = \xi_s\zeta_s\,(-)^{s-\sigma}\; a^{\dagger}(p;-\sigma,s), \tag{2.4.13}$$

$$(CPT)\, a^{\dagger}(p;\sigma,s)\,(CPT)^{-1} = (-)^{s-\sigma}\; b^{\dagger}(p;-\sigma,s). \tag{2.4.14}$$

The transformation properties for the particle annihilation operators are obtained from the above by an application of the Hermitian adjoint on both sides.

Analogous transformation properties hold for the antiparticle creation operators $b^{\dagger}(p;\sigma,s)$, albeit with independent phase factors as per the treatment in the previous section. These are given by

$$Cb^{\dagger}(p;\sigma,s)C^{-1} = \xi_s^{c*}\zeta_s^{c*}\, a^{\dagger}(p;\sigma,s), \tag{2.4.15}$$

$$Pb^{\dagger}(p;\sigma,s)P^{-1} = \xi_s^{c}\, b^{\dagger}(\mathscr{P}p;\sigma,s), \tag{2.4.16}$$

$$Tb^{\dagger}(p;\sigma,s)T^{-1} = \zeta_s^{c}\,(-)^{s-\sigma}\; b^{\dagger}(\mathscr{P}p;-\sigma,s). \tag{2.4.17}$$

Similarly, for the succession of discrete symmetries, we have

$$(CP)\, b^{\dagger}(p;\sigma,s)\,(CP)^{-1} = \zeta_s^{c*}\, a^{\dagger}(\mathscr{P}p;\sigma,s), \tag{2.4.18}$$

$$(CT)\, b^{\dagger}(p;\sigma,s)\,(CT)^{-1} = \xi_s^{c*}\,(-)^{s-\sigma}\; a^{\dagger}(\mathscr{P}p;-\sigma,s), \tag{2.4.19}$$

$$(PT)\, b^{\dagger}(p;\sigma,s)\,(PT)^{-1} = \xi_s^{c}\zeta_s^{c}\,(-)^{s-\sigma}\; b^{\dagger}(p;-\sigma,s), \tag{2.4.20}$$

$$(CPT)\, b^{\dagger}(p;\sigma,s)\,(CPT)^{-1} = (-)^{s-\sigma}\; a^{\dagger}(p;-\sigma,s). \tag{2.4.21}$$

Again, the transformation properties of $b(p;\sigma,s)$, the antiparticle annihilation operator, are obtained from (2.4.15)–(2.4.21) by the Hermitian adjoint.

2.5 The quantum field

In accordance with the treatment of Weinberg [64] the quantum field is here interpreted as a mere artifice for the construction of a Poincaré invariant S-matrix. By Poincaré invariance we mean invariance under the symmetries of the restricted Poincaré group $\mathcal{P}_+^\uparrow$, and, in field theories that include space-inversion and time-reversal, invariance also under the symmetries of the group of reflections $\mathcal{I}$. Weinberg cites three assumptions as the underlying considerations in the construction of the quantum field. We choose here to follow this approach and paraphrase Weinberg's three assumptions [64] as follows:

1. **Perturbation theory**

   The S-matrix can be calculated from the Dyson formula

   $$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \ldots dt_n T\left\{\mathcal{H}'(t_1)\ldots\mathcal{H}'(t_n)\right\}, \tag{2.5.1}$$

   where T denotes the time-ordered product. The Hamiltonian $\mathcal{H}$ is divided into a free-particle Hamiltonian, $\mathcal{H}_0$, and an interaction, $\mathcal{H}'$, such that $\mathcal{H} \equiv \mathcal{H}_0 + \mathcal{H}'$. Further, $\mathcal{H}'(t)$ is the interaction in the interaction representation; that is, $\mathcal{H}'(t) = \exp(+i\mathcal{H}_0 t)\mathcal{H}' \exp(-i\mathcal{H}_0 t)$.

2. **Poincaré invariance of the S-matrix**

   The S-matrix must be Poincaré invariant. To achieve this, it is sufficient to demand that $\mathcal{H}'(t)$ be given by

   $$\mathcal{H}'(t) = \int d^3x \mathscr{H}(\boldsymbol{x}, t), \tag{2.5.2}$$

   where the interaction density, $\mathscr{H}(\boldsymbol{x}, t)$, satisfies the following properties:

   a) It transforms as a scalar under the representations of the Poincaré group such that $U[\Lambda, a]\mathscr{H}(x)U^{-1}[\Lambda, a] = \mathscr{H}(\Lambda x + a)$.

   b) It commutes with itself at space-like separations; that is to say, for a space-like interval, $(x-y)^2 \equiv (x-y)^\mu (x-y)_\mu < 0$, $[\mathscr{H}(x), \mathscr{H}(y)] = 0$.

   Conditions (2.5.1) and (2.5.2) combine to yield

   $$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n T\left\{\mathscr{H}(x_1)\ldots\mathscr{H}(x_n)\right\}. \tag{2.5.3}$$

   This is manifestly Poincaré invariant, provided the step functions of the time-ordered product do not appear with a space-like argument. The latter requirement is met via condition 2b.

3. **Particle interpretation**

   The interaction density, $\mathscr{H}(x)$, must be constructed out of the creation and annihilation operators for the free particles described by $\mathcal{H}_0$.

Requirement 3, as per the quotation of Weinberg in the previous section, yields an S-matrix that automatically satisfies the cluster decomposition principle. Nevertheless, 3 also poses an immediate challenge when taken together with 2a, the requirement that the interaction density must transform as a scalar under the representations of the Poincaré group, because, from (2.4.6) and (2.4.7), the creation and annihilation operators do not transform as scalars. We must therefore devise a means whereby to put $a^\dagger(p;\sigma,s)$ and $a(p;\sigma,s)$ together in such a way that the resultant $\mathscr{H}(x)$ will transform in accordance with 2a. This is accomplished via the introduction of fields; for particle annihilation and particle creation, we introduce $\psi^+(x)$ and $\psi^-(x)$, respectively, given by linear combinations of particle annihilation and particle creation operators:

$$\psi_l^+(x) = \sum_{\sigma,s} \int d^3p\; u_l(x;p;\sigma,s)a(p;\sigma,s), \tag{2.5.4}$$

$$\psi_l^-(x) = \sum_{\sigma,s} \int d^3p\; v_l(x;p;\sigma,s)a^\dagger(p;\sigma,s), \tag{2.5.5}$$

where all three quantum numbers are summed over. The coefficient functions $u_l(x;p;\sigma,n)$ and $v_l(x;p;\sigma,n)$ are chosen such that, under the action of the representations of the Poincaré group, the field operators are multiplied by a position-independent matrix

$$U[\Lambda,a]\,\psi_l^\pm(x)\,U^{-1}[\Lambda,a] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda^{-1}]\,\psi_{\bar{l}}^\pm(\Lambda x + a). \tag{2.5.6}$$

Likewise for antiparticles, we introduce annihilation and creation fields:

$$\psi_l^{c+}(x) = \sum_{\sigma,s} \int d^3p\; u_l(x;p;\sigma,s)b(p;\sigma,s), \tag{2.5.7}$$

$$\psi_l^{c-}(x) = \sum_{\sigma,s} \int d^3p\; v_l(x;p;\sigma,s)b^\dagger(p;\sigma,s). \tag{2.5.8}$$

In accordance with an earlier postulate regarding the identical transform properties of particle and antiparticle states, and thereby particle and antiparticle creation and annihilation operators, the transformation properties of the antiparticle annihilation and creation fields must be identical to those of the corresponding particle annihilation and creation fields:

$$U[\Lambda,a]\,\psi_l^{c\pm}(x)\,U^{-1}[\Lambda,a] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda^{-1}]\,\psi_{\bar{l}}^{c\pm}(\Lambda x + a). \tag{2.5.9}$$

Accordingly, the coefficient functions employed above in the linear combination of antiparticle annihilation and creation operators are the same as those used in the corresponding linear combinations of particle annihilation and creation operators. Therefore, the position independent matrix, $D[\Lambda^{-1}]$, in (2.5.9) is the same as that in (2.5.6). The statement about the identical transformation properties is, of course, in relation to the unitary representations of the restricted Poincaré group. To the extent that the field theory under consideration is to respect space-inversion and time-reversal, we demand that the transformation properties (2.5.6) and (2.5.9) hold for the full Poincaré group. The case of charge-conjugation, and that of any symmetry involving the same, is somewhat different and will be addressed in Sec. 2.5.4. For the time being it will suffice to say that the charge-conjugated field will be equal to the original field up to complex-conjugation and multiplication by an appropriately chosen constant matrix.

We make one further remark on the position independent matrix. The possibility of having different matrices $D^{+}[\Lambda^{-1}]$ and $D^{-}[\Lambda^{-1}]$ for the annihilation and creation fields, respectively, is not considered. As we shall see below it will not suffice, on account of 2b, to consider $\psi^{+}(x)$ and $\psi^{-}(x)$ alone; instead, we must take linear combinations of these. A necessary and sufficient condition that will ensure that such linear combinations, with constant complex coefficients, will transform as prescribed in (2.5.6) is that the matrices $D^{+}[\Lambda^{-1}]$ and $D^{-}[\Lambda^{-1}]$ be the same.

Provided the interaction density is comprised of an even number of field components that create and destroy fermions along with any number of field components that create and destroy bosons, the remaining condition 2b will be satisfied if the components of the fields anticommute, '+', or commute, '−', at space-like separations:

$$\left[\psi_l^{+}(x), \psi_{\bar{l}}^{+}(y)\right]_{\pm} = \left[\psi_l^{-}(x), \psi_{\bar{l}}^{-}(y)\right]_{\pm} = \left[\psi_l^{+}(x), \psi_{\bar{l}}^{-}(y)\right]_{\pm} = 0, \tag{2.5.10}$$

$$\left[\psi_l^{c+}(x), \psi_{\bar{l}}^{c+}(y)\right]_{\pm} = \left[\psi_l^{c-}(x), \psi_{\bar{l}}^{c-}(y)\right]_{\pm} = \left[\psi_l^{c+}(x), \psi_{\bar{l}}^{c-}(y)\right]_{\pm} = 0, \tag{2.5.11}$$

for $(x-y)^2 < 0$. The cross terms must also vanish; this is guaranteed simply on account of the (anti-) commutation relations of the respective creation and annihilation operators given in (2.4.3). Also, by reason of (2.4.3), the first two (anti-) commutators in (2.5.10) and the first two (anti-) commutators in (2.5.11) automatically vanish for an arbitrary space-time interval. The remaining (anti-) commutators can be evaluated to read

$$\left[\psi_l^{+}(x), \psi_{\bar{l}}^{-}(y)\right]_{\pm} = \left[\psi_l^{c+}(x), \psi_{\bar{l}}^{c-}(y)\right]_{\pm} = \sum_{\sigma,s} \int d^3p\, u_l(x;p;\sigma,s) v_{\bar{l}}(y;p;\sigma,s).$$

The expression on the RHS fails to vanish in general, even for $(x-y)^2 < 0$. One might be inclined to circumvent this issue by attempting to construct the interaction density purely out of creation fields or purely out of annihilation fields; this, however, is not a viable approach because the resulting operator would fail to be Hermitian. As noted by Weinberg [40, p. 198], the only way to overcome the problem is to construct linear combinations of the form

$$\psi(x) \equiv \kappa\, \psi^{+}(x) + \lambda\, \psi^{c-}(x). \tag{2.5.12}$$

For future reference we write this out in full:

$$\psi_l(x) = \sum_{\sigma,s} \int d^3p\,(2\pi)^{-3/2} \Big[\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p;\sigma,s)a(p;\sigma,s) + \lambda\,\mathrm{e}^{+ip\cdot x}\,v_l(p;\sigma,s)b^\dagger(p;\sigma,s)\Big]. \tag{2.5.13}$$

By reason of the above transformation properties of the constituent fields, and bearing in mind the caveats regarding the reflections, $\psi(x)$ transforms according to

$$U[\Lambda,a]\,\psi_l(x)\,U^{-1}[\Lambda,a] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda^{-1}]\,\psi_{\bar{l}}(\Lambda x + a), \tag{2.5.14}$$

where $(\Lambda,a) \in \mathcal{P}_+^\uparrow$, or $a = 0$ and $\Lambda \in \mathcal{I}$. The coefficients κ and λ in (2.5.13) are complex numbers, chosen such that, along with a suitable choice of coefficient functions, we have

$$[\psi_l(x), \psi_{\bar{l}}(y)]_\pm = [\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_\pm = 0, \quad \text{for} \quad (x-y)^2 < 0, \tag{2.5.15}$$

where

$$\bar{\psi}_l(x) = \sum_{\sigma,s} \int d^3p\,(2\pi)^{-3/2} \Big[\kappa^*\,\mathrm{e}^{ip\cdot x}\,\bar{u}_l(p;\sigma,s)a^\dagger(p;\sigma,s) + \lambda^*\,\mathrm{e}^{-ip\cdot x}\,\bar{v}_l(p;\sigma,s)b(p;\sigma,s)\Big], \tag{2.5.16}$$

is the dual field expanded in terms of the dual coefficient functions

$$\bar{u}_l(p;\sigma,s) \equiv \sum_{\bar{l}} \eta_{l\bar{l}} u^*_{\bar{l}}(p;\sigma,s) \quad \text{and} \quad \bar{v}_l(p;\sigma,s) \equiv \sum_{\bar{l}} \eta_{l\bar{l}} v^*_{\bar{l}}(p;\sigma,s). \tag{2.5.17}$$

The matrix η, defined in detail in App. B.5 and illustrated by example in Sec. 3.3, can be thought of as the metric on the space of coefficient functions. Constraint (2.5.15) is equivalent to the corresponding expression given by Weinberg in [40, p. 198], so long as η is a constant matrix. Indeed, η is constant in all quantum field theories considered by Weinberg in [40, Ch. 5]; however, as is demonstrated in App. B.5, it is mathematically consistent to relax this assumption by allowing η to depend on the spin index s. In Ch. 3 and Ch. 4, we will show that the s dependence of the metric is crucial toward achieving consistency and unitarily at all energies without the need for regulator terms in theories of integer spin equal to or greater than one. We will take a closer look at (2.5.15) in Sec. 2.5.5. The (anti-)commutators will there be evaluated in terms of the coefficient functions.

In summary, the interaction density, $\mathscr{H}(x)$, is constructed out of $\psi(x)$ and its dual $\bar{\psi}(x)$. In order that $\mathscr{H}(x)$ will yield a Poincaré invariant S-matrix, the field and its dual must satisfy the following requirements. The field must transform under the unitary representations of the restricted Poincaré group as given in (2.5.14). To the extent that $\mathscr{H}(x)$ is to transform under the unitary representations of space-inversion and the antiunitarity representations of time-reversal according to 2a, the field must transform under these symmetries also according to (2.5.14). The transformation properties of the dual field are determined by those of $\psi(x)$, via the mapping between the respective coefficient functions given in (2.5.17); therefore, they place no further constraints upon the formalism, save the metric. This topic will be addressed in Sec. 3.3. Lastly, $\psi(x)$ and $\bar{\psi}(x)$ must satisfy the (anti-) commutation relations given in (2.5.15).

2.5.1 Representation of Poincaré group on space of coefficient functions

Let us now take a closer look at (2.5.14) and examine the properties of the position independent matrices $D[\Lambda^{-1}]$. Our objective will be to show that these matrices satisfy the composition rules of the underlying Minkowski space transformations.

Consider first the case where the symmetry group under consideration is the restricted Poincaré group. Here we may take two Minkowski space transformations Λ and $\bar{\Lambda}$, represented on the state space by unitary linear operators $U[\Lambda, a]$ and $U[\bar{\Lambda}, \bar{a}]$, and use the composition rule given (2.2.1) to determine the corresponding relation for the matrices $D[\Lambda]$ and $D[\bar{\Lambda}]$. Applying $U[\bar{\Lambda}, \bar{a}]$ to the transformed field (2.5.14), we obtain in index free notation

$$D[\Lambda^{-1}]D[\bar{\Lambda}^{-1}]\psi(\bar{\Lambda}\Lambda x + \bar{\Lambda}a + \bar{a}) = D[\Lambda^{-1}\bar{\Lambda}^{-1}]\psi(\bar{\Lambda}\Lambda x + \bar{\Lambda}a + \bar{a}). \qquad (2.5.18)$$

Now consider the case where $\Lambda, \bar{\Lambda} \in \mathcal{I}$, the group of reflections (2.1.6). The elements $\mathscr{P}$ and $\mathscr{T}$ each generate a subgroup of order two. In the case space-inversion, we found in Sec. 2.2.2 that it is represented on the state space by a unitary linear operator $U[\mathscr{P}]$. Applying two successive transformations to the field according to (2.5.14), and using the composition rule (2.3.76), we find

$$D[\mathscr{P}^{-1}]D[\mathscr{P}^{-1}]\psi(x) = D[\mathscr{P}^{-1}\mathscr{P}^{-1}]\psi(x). \qquad (2.5.19)$$

The Minkowski space symmetry of time-reversal, $\mathscr{T}$, was found in Sec. 2.2.2 to be represented on the state space by an antiunitarity antilinear operator $U[\mathscr{T}]$. Using (2.5.14) along with the composition rule (2.3.87) to apply time-reversal twice to $\psi(x)$, we obtain

$$D^*[\mathscr{T}^{-1}]D[\mathscr{T}^{-1}]\psi(x) = D[\mathscr{T}^{-1}\mathscr{T}^{-1}]\psi(x). \qquad (2.5.20)$$

Assuming for the time being that the coefficient functions $\boldsymbol{u}$ and $\boldsymbol{v}$ in the above expansion of the quantum field, (2.5.13), can be used to span a vector space of the appropriate dimensions, we deduce from (2.5.18), (2.5.19), and (2.5.20) that the following composition rules hold:

$$D[\Lambda]D[\bar{\Lambda}] = D[\Lambda\bar{\Lambda}], \quad \text{for} \quad \Lambda, \bar{\Lambda} \in \mathcal{L}_+^\uparrow, \qquad (2.5.21)$$

$$D[\Lambda]D[\bar{\Lambda}] = D[\Lambda\bar{\Lambda}], \quad \text{for} \quad \Lambda, \bar{\Lambda} \in \{1, \mathscr{P}\}, \qquad (2.5.22)$$

$$D[\Lambda]D[\bar{\Lambda}] = D[\Lambda\bar{\Lambda}], \quad \text{for} \quad \Lambda, \bar{\Lambda} \in \{1, \mathscr{T}\}, \qquad (2.5.23)$$

where in (2.5.23) we have made the simplifying assumption that the components of the matrix $D[\mathscr{T}]$ are strictly real; an obvious modification applies in the converse scenario. It is thus apparent that the matrices $D[\Lambda]$ satisfy the composition rules of the underlying Minkowski space transformations. Provided that $D[1]$ is proportional to the identity transformation on the space of coefficient functions, this shows that the matrices $D[\Lambda]$ furnish (projective) representations of the underlying Minkowski space symmetries: $\mathcal{L}_+^\uparrow$, $\{1, \mathscr{P}\}$, and $\{1, \mathscr{T}\}$. Respectively, these transformations are induced on the space of coefficient functions by the representations of the restricted Poincaré group and the representations of the reflections via the transformation property of the field as given in (2.5.14). Pleasing though it may be, this does not feature as a necessarily requirement in Weinberg's construction of quantum fields. Looking at [40, p. 192], we find that although Weinberg derives the composition rule for the matrices $D[\Lambda]$, he does not demand that the coefficient functions span the underlying four-dimensional vector space. The initial formulation of Weinberg's causal vector field [40, p. 207] provides a ready example of a quantum field theory in which the coefficient functions provide a basis for the representation $D[\Lambda]$. Further along in the same section, once the spin zero degree of freedom has been removed, a new basis is given in [40, p. 210]; this manifestly fails to span the underlying four-dimensional vector space.

Let us now explore the properties of $D[\Lambda]$ with $\Lambda \in \mathcal{L}_+^\uparrow$, under conjugation by $D[\bar{\Lambda}]$ with $\bar{\Lambda} \in \mathcal{I}$. Recalling the action of the representations of the reflections on the unitary representations of the restricted Poincaré group, given in (2.3.73) and (2.3.84), we have

$$U[\bar{\Lambda}, 0]U[\Lambda, a]U^{-1}[\bar{\Lambda}, 0] = U[\bar{\Lambda}\Lambda\bar{\Lambda}^{-1}, \bar{\Lambda}a]. \tag{2.5.24}$$

Rewriting (2.5.24) in the form

$$U[\bar{\Lambda}, 0]U[\Lambda, a] = U[\bar{\Lambda}\Lambda\bar{\Lambda}^{-1}, \bar{\Lambda}a]U[\bar{\Lambda}, 0], \tag{2.5.25}$$

and applying the LHS to the field $\psi(x)$, under invocation of (2.5.14), we obtain for $U[\bar{\Lambda}, 0]$ antiunitary

$$U[\bar{\Lambda}, 0]U[\Lambda, a]\psi(x)U^{-1}[\Lambda, a]U^{-1}[\bar{\Lambda}, 0] = D^*[\Lambda^{-1}]D[\bar{\Lambda}^{-1}]\psi(\bar{\Lambda}\Lambda x + \bar{\Lambda}a). \tag{2.5.26}$$

Repeating this in the case of a unitary $U[\bar{\Lambda}, 0]$, we have

$$U[\bar{\Lambda}, 0]U[\Lambda, a]\psi(x)U^{-1}[\Lambda, a]U^{-1}[\bar{\Lambda}, 0] = D[\Lambda^{-1}]D[\bar{\Lambda}^{-1}]\psi(\bar{\Lambda}\Lambda x + \bar{\Lambda}a). \tag{2.5.27}$$

Applying the RHS of (2.5.25) to $\psi(x)$ and noting that $\bar{\Lambda}\Lambda\bar{\Lambda}^{-1} \in \mathcal{L}_+^\uparrow$ for any $\bar{\Lambda} \in \mathcal{I}$ and $\Lambda \in \mathcal{L}_+^\uparrow$, one finds

$$\begin{aligned} U[\bar{\Lambda}\Lambda\bar{\Lambda}^{-1}, \bar{\Lambda}a]U[\bar{\Lambda}, 0]\psi(x)U^{-1}[\bar{\Lambda}, 0]U^{-1}[\bar{\Lambda}\Lambda\bar{\Lambda}^{-1}, \bar{\Lambda}a] \\ = D[\bar{\Lambda}]D[\bar{\Lambda}\Lambda^{-1}\bar{\Lambda}^{-1}]\psi(\bar{\Lambda}\Lambda x + \bar{\Lambda}a). \end{aligned} \tag{2.5.28}$$

We thus obtain

$$D[\bar{\Lambda}]D[\Lambda]D^{-1}[\bar{\Lambda}] = D[\bar{\Lambda}\Lambda\bar{\Lambda}^{-1}], \quad \text{for} \quad \Lambda \in \mathcal{L}_+^\uparrow,\ \bar{\Lambda} = \mathscr{P}, \tag{2.5.29}$$

$$D[\bar{\Lambda}]D[\Lambda]D^{-1}[\bar{\Lambda}] = D^*[\bar{\Lambda}\Lambda\bar{\Lambda}^{-1}], \quad \text{for} \quad \Lambda \in \mathcal{L}_+^\uparrow,\ \bar{\Lambda} = \mathscr{T}. \tag{2.5.30}$$

Equivalent to these and somewhat easier to apply toward the derivation of the matrices $D[\mathscr{P}]$ and $D[\mathscr{T}]$, within the context of a given representation $D[\Lambda]$ of the restricted Lorentz group, are the relations

$$D[\mathscr{P}]\mathscr{J}_i D^{-1}[\mathscr{P}] = +\mathscr{J}_i \quad \text{and} \quad D[\mathscr{P}]\mathscr{K}_i D^{-1}[\mathscr{P}] = -\mathscr{K}_i, \tag{2.5.31}$$

$$D[\mathscr{T}]\mathscr{J}_i D^{-1}[\mathscr{T}] = -\mathscr{J}_i^* \quad \text{and} \quad D[\mathscr{T}]\mathscr{K}_i D^{-1}[\mathscr{T}] = +\mathscr{K}_i^*, \tag{2.5.32}$$

where $\mathscr{J}_i$ and $\mathscr{K}_i$, $i \in \{x, y, z\}$, are the rotation and boost generators of $D[\Lambda]$, respectively. These expressions can be employed to derive the position independent matrices $D[\bar{\mathscr{P}}]$ and $D[\bar{\mathscr{T}}]$ uniquely up to a phase.

Before we proceed to the next section, it is worth noting that the representation of the restricted Lorentz group on the space of coefficient functions, as given above by the position independent matrices $D[\Lambda]$, with $\Lambda \in \mathcal{L}_+^\uparrow$, is non-unitary for all spins, with exception of the trivial case of a scalar field. This is because the matrices $D[\Lambda]$ furnish a finite-dimensional representation of $\mathcal{L}_+^\uparrow$ (including boosts), and as previously noted, $\mathcal{L}_+^\uparrow$ is a non-compact Lie group and therefore does not admit any non-trivial finite-dimensional unitary representations [40, p. 231]. The matrices $D[\Lambda]$ satisfy the relation

$$D^{-1}[\Lambda] = \eta\, D^\dagger[\Lambda]\, \eta^{-1}, \tag{2.5.33}$$

where η is the metric[3] of the space of coefficient functions. Weinberg uses the word "pseudounitarity" [40, p. 218] to describe the property (2.5.33). One might therefore say that the coefficient functions transform according to a pseudounitarity representation of the restricted Lorentz group.

2.5.2 Classification of quantum fields

There is further significance in the finite-dimensional pseudounitarity representations of the restricted Lorentz group: the underlying group structure can be used to provide a convenient classification of quantum fields. All the quantum fields studied in the present work pertain to one of the broad classes given in Tab. 2.2.

[3] The matrix η must not to be confused with the spacetime metric. They are proportional only in the special case of a vector field.

	Representation of $D[\Lambda]$	Corresponding Field
(a)	$(j, 0)$	Right Weyl type
(b)	$(0, j)$	Left Weyl type
(c)	$(j, 0) \oplus (0, j)$	Spinor fields
(d)	$(j, 0) \otimes (0, j)$ or equivalently (j, j)	Tensor fields

Table 2.2: Classification of quantum fields by their respective transformation properties. Here j may take integer or half-integer values. In (a)–(c), j represents the spin quantum number of the representation. In (d), $2j$ is the highest spin quantum number of the representation. A representation of type (d) for $2j \geq 1$ contains $2j + 1$ spin quantum numbers, s, where $s \in \{0, 1, \ldots, 2j\}$.

The representations of $D[\Lambda]$, where $\Lambda \in \mathcal{L}_{+}^{\uparrow}$, are defined in terms of $\mathscr{J}_i$ and $\mathscr{K}_i$, with $i \in \{x, y, z\}$, the generators of the Lie algebra of the restricted Lorentz group, as follows:

(a) The matrix representation $D[\Lambda]$ denoted by $(j, 0)$ is irreducible, having rotation generators, $\mathscr{J}_i$, given up to isomorphism by (2.3.30) with $s = j$; that is, $\mathscr{J}_i \cong J_i^{(j)}$. The generators of boost are given by $\mathscr{K}_i = -i\mathscr{J}_i$. The corresponding quantum field describes a particle with spin quantum number j.

(b) The matrix representation $D[\Lambda]$ denoted by $(0, j)$ is identical to $(j, 0)$ except for the boost generators which here are given by $\mathscr{K}_i = +i\mathscr{J}_i$. The corresponding quantum field describes a particle with spin quantum number j.

(c) The matrix representation $D[\Lambda]$ denoted by $(j, 0) \oplus (0, j)$ is reducible. Unlike cases (a) and (b), this representation admits a further symmetry, namely that of parity. Enlarging the symmetry group to include parity renders $(j, 0) \oplus (0, j)$ an irreducible representation. The underlying rotation generators, $\mathscr{J}_i$, are given up to isomorphism by a direct sum of the generators (2.3.30) with $s = j$; that is, $\mathscr{J}_i \cong J_i^{(j)} \oplus J_i^{(j)}$. The generators of boost are $\mathscr{K}_i \cong -iJ_i^{(j)} \oplus +iJ_i^{(j)}$. The corresponding quantum field describes a particle with spin quantum number j.

(d) The matrix representation $D[\Lambda]$ denoted by $(j, 0) \otimes (0, j)$ is irreducible, though the rotation subgroup is reducible. Looking at the rotation subgroup $D[\mathscr{R}]$ we find that it has $2j + 1$ spin degrees of freedom: $s \in \{0, 1, \ldots, 2j\}$. The underlying rotation generators, $\mathscr{J}_i$, are isomorphic to a direct sum containing $2j + 1$ generators $J_i^{(s)}$; that is, $\mathscr{J}_i \cong J_i^{(0)} \oplus J_i^{(1)} \oplus \ldots \oplus J_i^{(2j)}$. The generators of rotation and boost for $(j, 0) \otimes (0, j)$ can be computed by first constructing operators of rotation and boost for the representations $(j, 0)$ and $(j, 0)$, respectively, then calculating the tensor product of these operators and differentiating, in turn, by the six parameters associated with the six symmetry generators. Finally setting the parameters equal to zero and dividing by i gives the desired generators of $(j, 0) \otimes (0, j)$ up to isomorphism.

A detailed discussion of the underlying group theoretical structure of the (A, B) notation employed in Tab. 2.2 can be found in [40, Sec. 5.6].

2.5.3 Constraints on coefficient functions: continuous symmetries

Having given the general form of the annihilation and creation fields of particles in (2.5.4) and (2.5.5), and similarly for antiparticles in (2.5.7) and (2.5.8), all that remains is for us to give the explicit form of the expansion coefficients. We will now explore what is the explicit form of the constraints on the coefficient functions that follow from the above transformation property of the fields when taken together with the transformation properties of the underlying creation and annihilation operators. Explicit recourse in the present section to the quantum fields introduced in the opening remarks of Sec. 2.5 will be confined to $\psi^+(x)$ and $\psi^-(x)$; nonetheless, the constraints on the coefficient functions here to be derived apply equally to $\psi^{c+}(x)$ and $\psi^{c-}(x)$.

Applying a Poincaré transformation to the annihilation field $\psi^+(x)$, we have from (2.5.4) and (2.5.6)

$$\begin{aligned} U[\Lambda,a]\psi_l^+(x)U^{-1}[\Lambda,a] &= \sum_{\bar{l},\sigma,s}\int d^3p\; D_{l\bar{l}}[\Lambda^{-1}]u_{\bar{l}}(\Lambda x+a;p;\sigma,s)a(p;\sigma,s) \\ &= \sum_{\bar{l},\sigma,s}\int d^3p\frac{(\Lambda p)^0}{p^0}D_{l\bar{l}}[\Lambda^{-1}]u_{\bar{l}}(\Lambda x+a;\Lambda p;\sigma,s)a(\Lambda p;\sigma,s), \end{aligned}$$

where for the second equation we have used the Poincaré invariance of the measure d^3p/p^0 [40, p. 67]. Also, from (2.4.7) we have

$$\begin{aligned} U[\Lambda,a]\psi_l^+(x)U^{-1}[\Lambda,a] &= \sum_{\sigma',s}\int d^3p\; u_l(x;p;\sigma',s)U[\Lambda,a]\,a(p;\sigma',s)\,U^{-1}[\Lambda,a] \\ &= \sum_{\sigma',s}\int d^3p\; u_l(x;p;\sigma',s)\,\mathrm{e}^{-i\Lambda p\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma}D^{(s)}_{\sigma'\sigma}[W^{-1}(\Lambda,p)]\,a(\Lambda p;\sigma,s), \end{aligned}$$

where by (2.3.15), $W(\Lambda,p)\equiv L^{-1}(\Lambda p)\,\Lambda\,L(p)$, an element of the little group. Equating coefficients, we obtain the following constraint on the $\boldsymbol{u}$ coefficient functions:

$$\sum_{\bar{l}}D_{l\bar{l}}[\Lambda^{-1}]u_{\bar{l}}(\Lambda x+a;\Lambda p;\sigma,s)=\sum_{\sigma'}u_l(x;p;\sigma',s)\,\mathrm{e}^{-i\Lambda p\cdot a}\sqrt{\frac{p^0}{(\Lambda p)^0}}D^{(s)}_{\sigma'\sigma}[W^{-1}(\Lambda,p)].$$

Repeating the above, this time for the creation field $\psi^-(x)$, gives the corresponding constraint on the $\boldsymbol{v}$ coefficient functions:

$$\sum_{\bar{l}}D_{l\bar{l}}[\Lambda^{-1}]v_{\bar{l}}(\Lambda x+a;\Lambda p;\sigma,s)=\sum_{\sigma'}v_l(x;p;\sigma',s)\,\mathrm{e}^{+i\Lambda p\cdot a}\sqrt{\frac{p^0}{(\Lambda p)^0}}D^{(s)*}_{\sigma'\sigma}[W^{-1}(\Lambda,p)].$$

These may be rewritten more conveniently as

$$\sum_{\sigma} u_l(\Lambda x + a; \Lambda p; \sigma, s) D^{(s)}_{\sigma\sigma'}[W(\Lambda, p)] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda] u_{\bar{l}}(x; p; \sigma', s) \sqrt{\frac{p^0}{(\Lambda p)^0}}\, \mathrm{e}^{-i\Lambda p \cdot a}, \tag{2.5.34}$$

$$\sum_{\sigma} v_l(\Lambda x + a; \Lambda p; \sigma, s) D^{(s)*}_{\sigma\sigma'}[W(\Lambda, p)] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda] v_{\bar{l}}(x; p; \sigma', s) \sqrt{\frac{p^0}{(\Lambda p)^0}}\, \mathrm{e}^{+i\Lambda p \cdot a}. \tag{2.5.35}$$

These constraints on the coefficient functions are both necessary and sufficient in order to ensure that the ensuing creation and annihilation fields will transform appropriately under the action of the restricted Poincaré group. We now consider translations, boosts, and rotations in turn so as to determine the elements of the expansion coefficients in terms of a finite number of free parameters.

Translations

Consider a pure translation: $\Lambda = \mathbb{1}$ and a arbitrary. Given that $D[\Lambda]$ furnishes a representation of the restricted Lorentz group, $D[\mathbb{1}]$ is an identity matrix. Likewise, $W(\mathbb{1}, p) = L^{-1}(\mathbb{1}p)\mathbb{1}L(p) = \mathbb{1}$, such that $D^{(s)}[W(\mathbb{1}, p)]$ is an identity matrix. The constraints, (2.5.34) and (2.5.35), thus become

$$\boldsymbol{u}(x + a; p; \sigma, s) = \mathrm{e}^{-ip\cdot a}\, \boldsymbol{u}(x; p; \sigma, s),$$
$$\boldsymbol{v}(x + a; p; \sigma, s) = \mathrm{e}^{+ip\cdot a}\, \boldsymbol{v}(x; p; \sigma, s).$$

A simple choice of origin then yields

$$\boldsymbol{u}(x; p; \sigma, s) = \mathrm{e}^{-ip\cdot x}\, \boldsymbol{u}(0; p; \sigma, s),$$
$$\boldsymbol{v}(x; p; \sigma, s) = \mathrm{e}^{+ip\cdot x}\, \boldsymbol{v}(0; p; \sigma, s).$$

With the normalisation $\boldsymbol{u}(0; p; \sigma, s) = (2\pi)^{-3/2}\, \boldsymbol{u}(p; \sigma, s)$, and similarly for $\boldsymbol{v}$, we obtain

$$\boldsymbol{u}(x; p; \sigma, s) = (2\pi)^{-3/2}\, \mathrm{e}^{-ip\cdot x}\, \boldsymbol{u}(p; \sigma, s), \tag{2.5.36}$$

$$\boldsymbol{v}(x; p; \sigma, s) = (2\pi)^{-3/2}\, \mathrm{e}^{+ip\cdot x}\, \boldsymbol{v}(p; \sigma, s). \tag{2.5.37}$$

Field operators (2.5.4) and (2.5.5) thus take the familiar form akin to a three-dimensional Fourier transform:

$$\psi_l^+(x) = \sum_{\sigma,s} \int d^3p \, (2\pi)^{-3/2} \, \mathrm{e}^{-ip\cdot x} \, u_l(p;\sigma,s) a(p;\sigma,s), \tag{2.5.38}$$

$$\psi_l^-(x) = \sum_{\sigma,s} \int d^3p \, (2\pi)^{-3/2} \, \mathrm{e}^{+ip\cdot x} \, v_l(p;\sigma,s) a^\dagger(p;\sigma,s). \tag{2.5.39}$$

Before we proceed to consider boosts and rotations the constraints (2.5.34) and (2.5.35) may be simplified somewhat through the substitution of (2.5.36) and (2.5.37). We thus obtain

$$\sum_\sigma u_l(\Lambda p;\sigma,s) D^{(s)}_{\sigma\sigma'}[W(\Lambda,p)] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda] u_{\bar{l}}(p;\sigma',s) \sqrt{\frac{p^0}{(\Lambda p)^0}}, \tag{2.5.40}$$

$$\sum_\sigma v_l(\Lambda p;\sigma,s) D^{(s)*}_{\sigma\sigma'}[W(\Lambda,p)] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda] v_{\bar{l}}(p;\sigma',s) \sqrt{\frac{p^0}{(\Lambda p)^0}}. \tag{2.5.41}$$

These, along with (2.5.36) and (2.5.37), now constitute the fundamental constraints on the coefficient functions.

Boosts

For a pure boost we have $\Lambda = L(q)$. Choosing $p = k$, the momentum vector of a massive particle at rest, then gives $L(k) = \mathbb{1}$ such that the little group element becomes

$$W(L(q),k) = L^{-1}(L(q)k)L(q)L(k) = L^{-1}(q)L(q) = \mathbb{1}.$$

With these substitutions the constraint (2.5.40) and (2.5.41) read

$$u_l(q;\sigma,s) = \sqrt{\frac{m}{q^0}} \sum_{\bar{l}} D_{l\bar{l}}[L(q)] u_{\bar{l}}(k;\sigma,s), \tag{2.5.42}$$

$$v_l(q;\sigma,s) = \sqrt{\frac{m}{q^0}} \sum_{\bar{l}} D_{l\bar{l}}[L(q)] v_{\bar{l}}(k;\sigma,s). \tag{2.5.43}$$

Hence, the coefficient functions at arbitrary momentum q are obtained from those at rest via the boost matrix $D[L(q)]$ with the normalisation $\sqrt{m/q^0}$.

Rotations

Finally, for a pure rotation we have $\Lambda = \mathscr{R}$. Again, we choose $p = k$ such that

$$W(\mathscr{R},k) = L^{-1}(\mathscr{R}k)\mathscr{R}L(k) = L^{-1}(k)\mathscr{R}L(k) = \mathscr{R}.$$

The constraints (2.5.40) and (2.5.41) then read

$$\sum_{\sigma} u_l(k;\sigma,s) D^{(s)}_{\sigma\sigma'}[\mathscr{R}] = \sum_{\bar{l}} D_{l\bar{l}}[\mathscr{R}] u_{\bar{l}}(k;\sigma',s), \tag{2.5.44}$$

$$\sum_{\sigma} v_l(k;\sigma,s) D^{(s)*}_{\sigma\sigma'}[\mathscr{R}] = \sum_{\bar{l}} D_{l\bar{l}}[\mathscr{R}] v_{\bar{l}}(k;\sigma',s). \tag{2.5.45}$$

To simplify further, consider an infinitesimal rotation. It has non-zero components

$$\mathscr{R}_{ij} = \delta^i{}_j + \Omega_{ij}, \quad i,j \in \{1,2,3\}\,,$$

where $\delta^i{}_j = 1$ for $i = j$ and $\delta^i{}_j = 0$ for $i \neq j$; $\Omega_{ij} = -\Omega_{ji} \in \mathbb{R}$ and $\Omega_{ij}^2 \approx 0$. We then have

$$D^{(s)}_{\sigma\sigma'}[1+\Omega] = \delta_{\sigma\sigma'} + \frac{i}{2}\Omega_{jk}(J^{(s)}_{jk})_{\sigma\sigma'},$$

$$D_{l\bar{l}}[1+\Omega] = \delta^l{}_{\bar{l}} + \frac{i}{2}\Omega_{jk}(\mathscr{J}_{jk})_{l\bar{l}},$$

where $J^{(s)}_{kl}$ and $\mathscr{J}_{kl}$ each represent three generators of rotation. To minimise the use of indices, these will be represented as $J^{(s)}_{kl} = \frac{1}{2}i\epsilon_{klm}J^{(s)}_m$ and $\mathscr{J}_{kl} = \frac{1}{2}i\epsilon_{klm}\mathscr{J}_m$. Substitution into (2.5.44) and (2.5.45) thus yields

$$\sum_{\sigma} u_l(k;\sigma,s)(J^{(s)}_i)_{\sigma\sigma'} = \sum_{\bar{l}} (\mathscr{J}_i)_{l\bar{l}}\, u_{\bar{l}}(k;\sigma',s),$$

$$-\sum_{\sigma} v_l(k;\sigma,s)(J^{(s)*}_i)_{\sigma\sigma'} = \sum_{\bar{l}} (\mathscr{J}_i)_{l\bar{l}}\, v_{\bar{l}}(k;\sigma',s),$$

or alternatively

$$\sum_{\sigma} u_l(k;\sigma,s)\boldsymbol{J}^{(s)}_{\sigma\sigma'} = \sum_{\bar{l}} \boldsymbol{\mathscr{J}}_{l\bar{l}}\, u_{\bar{l}}(k;\sigma',s), \tag{2.5.46}$$

$$-\sum_{\sigma} v_l(k;\sigma,s)\boldsymbol{J}^{(s)*}_{\sigma\sigma'} = \sum_{\bar{l}} \boldsymbol{\mathscr{J}}_{l\bar{l}}\, v_{\bar{l}}(k;\sigma',s). \tag{2.5.47}$$

We know from the results of Sec. 2.3.1 that the three rotation generators that constitute $\boldsymbol{J}^{(s)}_{\sigma\sigma'}$ are simply the standard angular momentum matrices. Hence

$$(\boldsymbol{J}^{(s)})_{\sigma\sigma'} \equiv (J^{(s)}_i)_{\sigma'\sigma} \equiv \langle\sigma', s|J_i|\sigma, s\rangle,$$

with $i \in \{x,y,z\}$. These are given explicitly in (2.3.30).

It will prove computationally favourable to express constraints (2.5.46) and (2.5.47) in index free form. In doing so, we shall consider the following two distinct cases:

(i) The free-field theory involves a single spin degree of freedom.

(ii) The free-field theory involves $j+1$ spin degrees of freedom.

In the construction of a quantum field á la Weinberg the spin content of the theory is established at the outset in the definition of the state space. For the remainder of this section, we will explore constraints (2.5.47) and (2.5.48) in each of these cases in turn.

In case (i), the state space quantum number s takes a single value $s=j$. In agreement with the expressions given by Weinberg [40, p. 196], the constraints (2.5.46) and (2.5.47) then read

$$\sum_{\sigma} u_l(k;\sigma)\boldsymbol{J}^{(j)}_{\sigma\sigma'} = \sum_{\bar{l}} \mathscr{J}_{l\bar{l}}\, u_{\bar{l}}(k;\sigma'), \tag{2.5.48}$$

$$-\sum_{\sigma} v_l(k;\sigma)\boldsymbol{J}^{(j)*}_{\sigma\sigma'} = \sum_{\bar{l}} \mathscr{J}_{l\bar{l}}\, v_{\bar{l}}(k;\sigma'). \tag{2.5.49}$$

Here the spin projection index ranges over the values $\sigma \in \{j, j-1, \dots, 1-j, -j\}$; the spin index on the expansion coefficients has been omitted because it is redundant in the case at hand. The corresponding quantum fields may be chosen from among any one of the classes (a)–(d) of Tab. 2.2. Constraints (2.5.48) and (2.5.49) are now expressed in matrix form via the definitions

$$\boldsymbol{J} \equiv \left(\boldsymbol{J}^{(j)}_{\sigma\sigma'}\right), \qquad U \equiv (U_{l\sigma}) \equiv (u_l(k;\sigma)),$$

$$\mathscr{J} \equiv (\mathscr{J}_{l\bar{l}}), \qquad V \equiv (V_{l\sigma}) \equiv (v_l(k;\sigma)),$$

yielding

$$U\boldsymbol{J} = \mathscr{J} U \quad \text{and} \quad -V\boldsymbol{J}^* = \mathscr{J} V. \tag{2.5.50}$$

A useful matrix in this context is the Wigner time-reversal operator Θ. It is defined by the relation

$$\Theta \boldsymbol{J} \Theta^{-1} = -\boldsymbol{J}^*, \tag{2.5.51}$$

where $\boldsymbol{J}$ may be any rotation generator. With this, (2.5.50) may be written as

$$U\boldsymbol{J} = \mathscr{J} U \quad \text{and} \quad V\Theta\boldsymbol{J} = \mathscr{J} V\Theta. \tag{2.5.52}$$

The constraint equations for case (i) are now in their final form.

It is worth noting that the matrices U and V turn out to be square only if the dimensions of the matrices J_i are the same as those of $\mathscr{J}_i$. Consider, for instance, the case of spin $j=1/2$. Here the matrices J_i pertaining to the state space are proportional to the 2×2 Pauli matrices. The quantum field may be of Weyl type, in which case the generators $\mathscr{J}_i$ are those of the $(1/2,0)$ representation; these are isomorphic to J_i and thereby also 2×2, thus rendering U and V square. Alternatively, the field could be a spinor field, the Dirac field, in which case the generators $\mathscr{J}_i$ are of the $(1/2,0)\oplus(0,1/2)$ representation; consequently, they are proportional to $J_i \oplus J_i$, and thereby 4×4. Hence, in this case, U and V are 4×2.

In case (ii), we wish to construct a free-field theory that involves more than one spin degree of freedom; specifically, one in which s takes $j+1$ integer values from $s=0$

to $s = j$. Here we must define a state space with these quantum numbers, a possibility explicitly accounted for in Sec. 2.3. Correspondingly the representation $D[\Lambda]$ pertaining to the quantum field must also have these $j+1$ spin degrees of freedom. Among the classes considered in Tab. 2.2, (d) is the only one that allows for this possibility. In fact, the $(j/2, j/2)$ representation intrinsically contains the $j+1$ spin degrees of freedom from $s=0$ to $s=j$ because the underlying rotation generators are isomorphic to $J_i^{(0)} \oplus J_i^{(1)} \oplus \cdots \oplus J_i^{(j)}$.

As for the constraint (2.5.46) and (2.5.47) s takes all integer values from $s=0$ to $s=j$. We have two sets of three constraint equations for each value of s. This can be simplified through the introduction of a block diagonal matrix comprised of irreducible representations of the rotation group, one for each value of s. We thus define

$$\boldsymbol{J} \equiv (\boldsymbol{J}_{\alpha\alpha'}) \equiv \left(\boldsymbol{J}^{(0)} \oplus \boldsymbol{J}^{(1)} \oplus \cdots \oplus \boldsymbol{J}^{(j)}\right). \tag{2.5.53}$$

Here α takes the values $\{1, \ldots, n\}$ where

$$n = \sum_{s=0}^{s=j} (2s+1)\,. \tag{2.5.54}$$

With three further definitions

$$\boldsymbol{\mathscr{J}} \equiv (\boldsymbol{\mathscr{J}}_{l\bar{l}})\,, \tag{2.5.55}$$

$$U \equiv (U_{l\alpha}) \equiv (u_l(k;\sigma,s))\,, \tag{2.5.56}$$

$$V \equiv (V_{l\alpha}) \equiv (v_l(k;\sigma,s))\,, \tag{2.5.57}$$

the constraints (2.5.46) and (2.5.47) may be written in matrix notation as

$$U\boldsymbol{J} = \boldsymbol{\mathscr{J}}U \quad \text{and} \quad -V\boldsymbol{J}^* = \boldsymbol{\mathscr{J}}V, \tag{2.5.58}$$

or, again invoking the Wigner time-reversal operator,

$$U\boldsymbol{J} = \boldsymbol{\mathscr{J}}U \quad \text{and} \quad V\Theta\boldsymbol{J} = \boldsymbol{\mathscr{J}}V\Theta. \tag{2.5.59}$$

In this case, U and V are indeed square because J_i and $\mathscr{J}_i$ are of the same dimensions. For $j=0$ they are 1×1, for $j=1$ they are 4×4, and for $j=2$ they are 9×9; the latter two will be the topics of Ch. 3 and Ch. 4, respectively, where the matrices U and ΘV will be derived as similarity transformations between the generators $\boldsymbol{\mathscr{J}}$ of $(j/2, j/2)$ and the generators $\boldsymbol{J}$ defined in (2.5.53).

With this, the constraint equations arising from the respective transformation properties of the state space and the quantum field under the rotation subgroup of the Lorentz group are in their final form. We will demonstrate in the next two chapters how the here derived constraints can be employed to determine the components of the coefficient functions at rest in terms of a finite number of free parameters. Once these are established, the coefficient functions are given at arbitrary momenta by (2.5.42) and (2.5.43). Further constraints arising from the discrete symmetries will be derived in the next section.

2.5.4 Constraints on coefficient functions: discrete symmetries

Having derived constraints on the coefficient functions by demanding that the quantum field exhibit appropriate transformation properties under the action of the unitary representations of the restricted Poincaré group, we now turn to the discrete symmetries. We shall proceed in much the same way as we did in the previous section; however, instead of using only $\psi^+(x)$ and $\psi^-(x)$ to derive constraints on the coefficient functions, it will be necessary here to consider $\psi(x)$ and $\bar{\psi}(x)$. This is because, in Sec. 2.3.2, we allowed for the possibility that the phases incurred by the particle states under the representations of the discrete symmetries may be chosen independently from those incurred by the antiparticle states under the representations of the discrete symmetries. These phases will come to bear via the creation and annihilation operators in $\psi(x)$ and $\bar{\psi}(x)$.

We begin by recalling the transformation properties of $\psi(x)$ under the representations of the reflections, as obtained in Sec. 2.5 from the requirement that the S-matrix be invariant under space-inversion and time-reversal. From (2.5.14), we have

$$U[\Lambda]\psi_l(x)U^{-1}[\Lambda] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda^{-1}]\psi_{\bar{l}}(\Lambda x), \tag{2.5.60}$$

where $U[\Lambda] \equiv U[\Lambda, 0]$ and $\Lambda \in \mathcal{I}$. In keeping with the notation of Sec. 2.3.2 we here use $P \equiv U[\mathscr{P}]$ and $T \equiv U[\mathscr{T}]$. It will prove convenient to expand (2.5.60) explicitly in terms of the creation and annihilation operators and their respective coefficient functions. From (2.5.13), we obtain

$$\begin{aligned} U[\Lambda]\psi_l(x)U^{-1}[\Lambda] = \sum_{\bar{l}} \sum_{\sigma,s} \int d^3p\,(2\pi)^{-3/2}\big[\kappa\, \mathrm{e}^{-ip\cdot\Lambda x}\, D_{l\bar{l}}[\Lambda^{-1}]u_{\bar{l}}(p;\sigma,s)a(p;\sigma,s) \\ +\lambda\, \mathrm{e}^{+ip\cdot\Lambda x}\, D_{l\bar{l}}[\Lambda^{-1}]v_{\bar{l}}(p;\sigma,s)b^\dagger(p;\sigma,s)\big]. \end{aligned} \tag{2.5.61}$$

We now proceed to derive constraints on the coefficient functions that result from space-inversion and time-reversal. This will be achieved by first using the transformation properties of the creation and annihilation operators, as derived in Sec. 2.4, and then manipulating the resulting expression for $U[\Lambda]\psi(x)U^{-1}[\Lambda]$ so as to admit direct comparison with (2.5.61). The transformation properties of the coefficient functions under $D[\Lambda^{-1}]$ can then be read off. Charge-conjugation requires further discussion; this will be addressed after space-inversion and time-reversal.

Space-inversion

Using the transformation properties of the state space operators for particle annihilation and antiparticle creation, given respectively by (2.4.9) and (2.4.16), under the unitary representation of space-inversion, the transformed field may be written as

$$
\begin{aligned}
P\psi_l(x)P^{-1} &= \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\left[\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p;\sigma,s)Pa(p;\sigma,s)P^{-1}\right.\\
&\qquad\qquad\left.+\lambda\,\mathrm{e}^{+ip\cdot x}\,v_l(p;\sigma,s)Pb^\dagger(p;\sigma,s)P^{-1}\right]\\
&= \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\left[\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p;\sigma,s)\xi_s^* a(\mathscr{P}p;\sigma,s)\right.\\
&\qquad\qquad\left.+\lambda\,\mathrm{e}^{+ip\cdot x}\,v_l(p;\sigma,s)\xi_s^c b^\dagger(\mathscr{P}p;\sigma,s)\right]\\
&= \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\Big[\kappa\,\mathrm{e}^{-ip\cdot\mathscr{P}x}\,\xi_s^* u_l(\mathscr{P}p;\sigma,s)a(p;\sigma,s)\\
&\qquad\qquad+\lambda\,\mathrm{e}^{+ip\cdot\mathscr{P}x}\,\xi_s^c v_l(\mathscr{P}p;\sigma,s)b^\dagger(p;\sigma,s)\Big]\,.
\end{aligned}
$$

Term by term comparison with (2.5.61) immediately gives the following property for the coefficient functions under multiplication by $D[\mathscr{P}^{-1}]$:

$$\sum_{\bar{l}} D_{l\bar{l}}[\mathscr{P}^{-1}]u_{\bar{l}}(p;\sigma,s) = \xi_s^* u_l(\mathscr{P}p;\sigma,s), \tag{2.5.62}$$

$$\sum_{\bar{l}} D_{l\bar{l}}[\mathscr{P}^{-1}]v_{\bar{l}}(p;\sigma,s) = \xi_s^c v_l(\mathscr{P}p;\sigma,s). \tag{2.5.63}$$

The matrix $D[\mathscr{P}^{-1}]$ must be non-singular and is derived via the relations (2.5.31). Furthermore, by (2.5.29), it satisfies the composition rule

$$D[\mathscr{P}^{-1}]\,D[L(p)]\,D^{-1}[\mathscr{P}^{-1}] = D[L(\mathscr{P}p)]. \tag{2.5.64}$$

Consequently, the constraints (2.5.62) and (2.5.63) are equivalent to

$$\sum_{\bar{l}} D_{l\bar{l}}[\mathscr{P}^{-1}]u_{\bar{l}}(k;\sigma,s) = \xi_s^* u_l(k;\sigma,s), \tag{2.5.65}$$

$$\sum_{\bar{l}} D_{l\bar{l}}[\mathscr{P}^{-1}]v_{\bar{l}}(k;\sigma,s) = \xi_s^c v_l(k;\sigma,s). \tag{2.5.66}$$

For future reference, provided that (2.5.65) and (2.5.66) are met, the parity transformed field reads

$$P\psi_l(x)P^{-1} = \sum_{\bar{l}} D_{l\bar{l}}[\mathscr{P}^{-1}]\psi_{\bar{l}}(\mathscr{P}x), \tag{2.5.67}$$

as desired.

Time-reversal

From the transformation properties of the creation and annihilation operators given in (2.4.10) and (2.4.17), and recalling that time-reversal is represented on the state space by an antiunitary antilinear operator, we have

$$\begin{aligned}
T\psi_l(x)T^{-1} &= \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\left[\kappa^*\,\mathrm{e}^{ip\cdot x}\,u_l^*(p;\sigma,s)Ta(p;\sigma,s)T^{-1}\right.\\
&\qquad\qquad\left.+\lambda^*\,\mathrm{e}^{-ip\cdot x}\,v_l^*(p;\sigma,s)Tb^\dagger(p;\sigma,s)T^{-1}\right]\\
&= \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\left[\kappa^*\,\mathrm{e}^{ip\cdot x}\,u_l^*(p;\sigma,s)\zeta_s^*(-)^{s-\sigma}a(\mathscr{P}p;-\sigma,s)\right.\\
&\qquad\qquad\left.+\lambda^*\,\mathrm{e}^{-ip\cdot x}\,v_l^*(p;\sigma,s)\zeta_s^c(-)^{s-\sigma}b^\dagger(\mathscr{P}p;-\sigma,s)\right]\\
&= \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\left[\kappa\,\mathrm{e}^{ip\cdot\mathscr{P}x}\,\frac{\kappa^*}{\kappa}\zeta_s^*(-)^{s+\sigma}u_l^*(\mathscr{P}p;-\sigma,s)a(p;\sigma,s)\right.\\
&\qquad\qquad\left.+\lambda\,\mathrm{e}^{-ip\cdot\mathscr{P}x}\,\frac{\lambda^*}{\lambda}\zeta_s^c(-)^{s+\sigma}v_l^*(\mathscr{P}p;-\sigma,s)b^\dagger(p;\sigma,s)\right].
\end{aligned}$$

Comparison with (2.5.61) yields

$$\sum_{\bar{l}} D_{l\bar{l}}[\mathscr{T}^{-1}]u_{\bar{l}}(p;\sigma,s) = \frac{\kappa^*}{\kappa}\zeta_s^*(-)^{s+\sigma}u_l^*(\mathscr{P}p;-\sigma,s), \tag{2.5.68}$$

$$\sum_{\bar{l}} D_{l\bar{l}}[\mathscr{T}^{-1}]v_{\bar{l}}(p;\sigma,s) = \frac{\lambda^*}{\lambda}\zeta_s^c(-)^{s+\sigma}v_l^*(\mathscr{P}p;-\sigma,s), \tag{2.5.69}$$

where $D[\mathscr{T}^{-1}]$ is a non-singular matrix derived via (2.5.32). By (2.5.30), it satisfies

$$D[\mathscr{T}^{-1}]\,D[L(p)]\,D^{-1}[\mathscr{T}^{-1}] = D^*[L(\mathscr{P}p)]. \tag{2.5.70}$$

Consequently, (2.5.68) and (2.5.69) are equivalent to

$$\sum_{\bar{l}} D_{l\bar{l}}[\mathscr{T}^{-1}]u_{\bar{l}}(k;\sigma,s) = \frac{\kappa^*}{\kappa}\zeta_s^*(-)^{s+\sigma}u_l^*(k;-\sigma,s), \tag{2.5.71}$$

$$\sum_{\bar{l}} D_{l\bar{l}}[\mathscr{T}^{-1}]v_{\bar{l}}(k;\sigma,s) = \frac{\lambda^*}{\lambda}\zeta_s^c(-)^{s+\sigma}v_l^*(k;-\sigma,s). \tag{2.5.72}$$

For future reference, provided that (2.5.71) and (2.5.72) are met, the transformation of the field under time-reversal reads

$$T\psi_l(x)T^{-1} = D_{l\bar{l}}[\mathscr{T}^{-1}]\psi_{\bar{l}}(\mathscr{T}x), \tag{2.5.73}$$

as desired.

Charge-conjugation

Before we directly address the question of charge-conjugation, let us take a brief look at Weinberg's treatment of discrete symmetries as given in [40, Ch. 5]. Weinberg emphasises that the transformations must be such that the transformed fields commute with the original fields at space-like separations [40, p. 205] [40, p. 213] [40, p. 225]. This is because for an interacting theory, in which the Hamiltonian is given by the free-particle Hamiltonian plus an interaction, there may appear in the interaction not only fields and dual fields but also discrete symmetry transformed fields and dual fields. In order for the corresponding interaction density to commute with itself at space-like separations, it is sufficient, along with the previous requirement that the field must commute with itself and with its dual at space-like separations, to demand

$$[P\psi_l(x)P^{-1},\psi_{\bar{l}}(y)]_\pm = 0, \qquad [P\psi_l(x)P^{-1},\bar{\psi}_{\bar{l}}(y)]_\pm = 0, \tag{2.5.74}$$

$$[T\psi_l(x)T^{-1},\psi_{\bar{l}}(y)]_\pm = 0, \qquad [T\psi_l(x)T^{-1},\bar{\psi}_{\bar{l}}(y)]_\pm = 0, \tag{2.5.75}$$

$$[C\psi_l(x)C^{-1},\psi_{\bar{l}}(y)]_\pm = 0, \qquad [C\psi_l(x)C^{-1},\bar{\psi}_{\bar{l}}(y)]_\pm = 0, \tag{2.5.76}$$

for $(x-y)^2 < 0$. It is easy to see that the constraints imposed in the preceding two sections for space-inversion and time-reversal are sufficient to ensure that (2.5.74) and (2.5.75) are met, provided $\psi(x)$ and $\bar{\psi}(x)$ satisfy (2.5.74). To illustrate, consider the case of space-inversion. Expanding the (anti-) commutators in (2.5.74) in terms of the transformed field, as given in (2.5.67), we obtain

$$[P\psi_l(x)P^{-1},\psi_{\bar{l}}(y)]_\pm = \sum_k D_{lk}[\mathscr{P}^{-1}][\psi_k(\mathscr{P}x),\psi_{\bar{l}}(y)]_\pm, \tag{2.5.77}$$

$$[P\psi_l(x)P^{-1},\bar{\psi}_{\bar{l}}(y)]_\pm = \sum_k D_{lk}[\mathscr{P}^{-1}][\psi_k(\mathscr{P}x),\bar{\psi}_{\bar{l}}(y)]_\pm. \tag{2.5.78}$$

Clearly, if x and y are space-like separated, then so are the transformed coordinates $\mathscr{P}x$ and y. Therefore, if (2.5.15) is satisfied, that is, if the field (anti-) commutes with itself and with the dual field at space-like separations, then (2.5.77) and (2.5.78) will automatically vanish and thereby satisfy (2.5.74). The case of time-reversal is similar.

The question thus arises, what is the simplest way to express $C\psi(x)C^{-1}$ in terms of $\psi(x)$ such that the constraints in (2.5.76) are satisfied? Bearing in mind that this must be consistent also with the transformation properties of the creation and annihilation operators given in (2.4.8) and (2.4.15), we postulate, in keeping with Weinberg [40, Ch. 5] and in close analogy to (2.5.60), that the transformed field must take the form

$$C\psi_l(x)C^{-1} = \sum_{\bar{l}} A_{l\bar{l}}\psi^*_{\bar{l}}(x), \tag{2.5.79}$$

where A is a yet to be determined complex valued square non-singular constant matrix of dimensions $l \times \bar{l}$. To verify that this expression satisfies the stated requirements, we begin by checking its consistency with the previously derived transformation properties of the creation and annihilation operators. The RHS of (2.5.79), can be expanded via (2.5.13), to read

$$\sum_{\bar{l}} A_{l\bar{l}}\psi^*_{\bar{l}}(x) = \sum_{\bar{l}}\sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\Big[\kappa^*\,\mathrm{e}^{+ip\cdot x}\,A_{l\bar{l}}u^*_{\bar{l}}(p;\sigma,s)a^\dagger(p;\sigma,s)$$
$$+\lambda^*\,\mathrm{e}^{-ip\cdot x}\,A_{l\bar{l}}v^*_{\bar{l}}(p;\sigma,s)b(p;\sigma,s)\Big]. \quad (2.5.80)$$

Conversely, recalling that charge-conjugation is represented on the state space by a unitary linear operator and using the transformation properties of $a(p;\sigma,s)$ and $b^\dagger(p;\sigma,s)$ as given in (2.4.8) and (2.4.15), we may write

$$C\psi_l(x)C^{-1} = \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\Big[\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p;\sigma,s)Ca(p;\sigma,s)C^{-1}$$
$$+\lambda\,\mathrm{e}^{+ip\cdot x}\,v_l(p;\sigma,s)Cb^\dagger(p;\sigma,s)C^{-1}\Big]$$
$$= \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\Big[\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p;\sigma,s)\xi_s\zeta_s\,b(p;\sigma,s)$$
$$+\lambda\,\mathrm{e}^{+ip\cdot x}\,v_l(p;\sigma,s)\xi^{c*}_s\zeta^{c*}_s\,a^\dagger(p;\sigma,s)\Big]$$
$$= \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\Big[\kappa^*\,\mathrm{e}^{+ip\cdot x}\,\frac{\lambda}{\kappa^*}\xi^{c*}_s\zeta^{c*}_s v_l(p;\sigma,s)a^\dagger(p;\sigma,s)$$
$$+\lambda^*\,\mathrm{e}^{-ip\cdot x}\,\frac{\kappa}{\lambda^*}\xi_s\zeta_s\,u_l(p;\sigma,s)b(p;\sigma,s)\Big].$$

The rearrangement performed in the last equation now allows direct term by term comparison with (2.5.80). Consistency thus requires that

$$\sum_{\bar{l}} A_{l\bar{l}}u^*_{\bar{l}}(p;\sigma,s) = \frac{\lambda}{\kappa^*}\xi^{c*}_s\zeta^{c*}_s v_l(p;\sigma,s), \quad (2.5.81)$$

$$\sum_{\bar{l}} A_{l\bar{l}}v^*_{\bar{l}}(p;\sigma,s) = \frac{\kappa}{\lambda^*}\xi_s\zeta_s\,u_l(p;\sigma,s). \quad (2.5.82)$$

Provided there exists a suitable matrix A such that (2.5.81) and (2.5.82) can be met, the postulated transformation property (2.5.79) is consistent with the transformation properties of the creation and annihilation operators.

We now check the second requirement, namely that (2.5.79) must yield a transformed field that satisfies (2.5.76). Substituting (2.5.79) into the LHS of (2.5.76), one obtains

$$[C\psi_l(x)C^{-1},\psi_{\bar{l}}(y)]_\pm = \sum_k A_{lk}[\psi^*_k(x),\psi_{\bar{l}}(y)]_\pm, \quad (2.5.83)$$

$$[C\psi_l(x)C^{-1},\bar{\psi}_{\bar{l}}(y)]_\pm = \sum_k A_{lk}[\psi^*_k(x),\bar{\psi}_{\bar{l}}(y)]_\pm. \quad (2.5.84)$$

Note that if the metric η is a constant matrix, as is the case in all the field theories considered by Weinberg in [40, Ch. 5], then we can rewrite (2.5.83) and (2.5.84) as

$$[C\psi_l(x)C^{-1}, \psi_{\bar{l}}(y)]_\pm = \sum_k A_{lk}[\psi_k(x), \psi^*_{\bar{l}}(y)]^*_\pm, \tag{2.5.85}$$

$$[C\psi_l(x)C^{-1}, \bar{\psi}_{\bar{l}}(y)]_\pm = \sum_{k\bar{k}} A_{lk}\eta_{\bar{l}\bar{k}}[\psi_k(x), \psi_{\bar{k}}(y)]^*_\pm. \tag{2.5.86}$$

These are manifestly proportional to the complex conjugate of the (anti-) commutators in (2.5.15). Hence, provided (2.5.15) is satisfied, the demand that $C\psi(x)C^{-1}$ be of the form postulated in (2.5.79) immediately guarantees that the constraints in (2.5.76) are met.

Unfortunately, the matter is not quite so simple if we relax the assumption about the metric being constant. In Ch. 3 and Ch. 4 we will construct theories in which η has an explicit functional dependence on the spin index s. Expanding the RHS of (2.5.83) in terms of the coefficient functions, and demanding that the resulting expression must vanish at space-like separations in accordance with (2.5.76), we find, having suppressed the column index,

$$\begin{aligned}\sum_{\sigma,s}\int \frac{d^3p}{(2\pi)^3}\Big[|\kappa|^2\, \boldsymbol{u}(p;\sigma,s)\boldsymbol{u}^\dagger(p;\sigma,s)\,\mathrm{e}^{-ip\cdot(x-y)}&\\ \pm|\lambda|^2\,\boldsymbol{v}(p;\sigma,s)\boldsymbol{v}^\dagger(p;\sigma,s)\,\mathrm{e}^{+ip\cdot(x-y)}\Big] &= 0,\end{aligned} \tag{2.5.87}$$

for $(x-y)^2 < 0$. Likewise, expanding (2.5.84), we find this vanishes on account of the (anti-) commutation relations of the creation and annihilation operators, save the case of an identical antiparticle. In the case of an identical antiparticle, the demand that (2.5.84) must vanish at space-like separations is equivalent to

$$\begin{aligned}\sum_{\sigma,s}\int \frac{d^3p}{(2\pi)^3}\Big[\boldsymbol{u}(p;\sigma,s)\boldsymbol{v}^T(p;\sigma,s)\eta(s)\,\mathrm{e}^{-ip\cdot(x-y)}&\\ \pm\boldsymbol{v}(p;\sigma,s)\boldsymbol{u}^T(p;\sigma,s)\eta(s)\,\mathrm{e}^{+ip\cdot(x-y)}\Big] &= 0,\end{aligned} \tag{2.5.88}$$

for $(x-y)^2 < 0$. Although these results would seem like bad news in that they appear to indicate that (2.5.79) fails to guarantee (2.5.76) in the case of an s-dependent metric, we will find that, at least for the two field theories constructed respectively in Ch. 3 and Ch. 4, (2.5.87) and (2.5.88) are satisfied automatically once all the other constraints have been imposed. We thus take (2.5.79) as the desired transformation property of the field under the unitary charge-conjugation operator C, keeping (2.5.87) and (2.5.88) as further constraints to be checked once (2.5.79) has been imposed.

In order to complete the above description, we must determine how the charge-conjugation matrix A relates to $D[\Lambda]$, for $\Lambda \in \mathcal{L}^\uparrow_+$. We shall seek expressions that will allow us to derive A by considering its properties with respect to the generators of rotation and boost that underlie $D[\Lambda]$, akin to those derived in Sec. 2.5.1 for space-inversion and time-reversal. Invoking the transformation property of the field under C, as in (2.5.79), followed by its transformation property under $U[\Lambda, a]$, as in (2.5.14), we obtain

$$U[\Lambda, a]C\psi(x)C^{-1}U^{-1}[\Lambda, a]C^{-1} = AU[\Lambda, a]\psi^*(x)U^{-1}[\Lambda, a] = AD^*[\Lambda^{-1}]\psi^*(\Lambda x+a).$$

Now using (2.5.14) first, and then (2.5.79), we find

$$CU[\Lambda, a]\psi(x)U^{-1}[\Lambda, a]C^{-1} = CD[\Lambda^{-1}]\psi(\Lambda x + a)C^{-1} = D[\Lambda^{-1}]A\psi^*(\Lambda x + a).$$

Recalling from (2.3.89) that $CU[\Lambda, a] = U[\Lambda, a]C$, and assuming, as we did in Sec. 2.5.1, that the coefficient functions can be used to span a vector space of the appropriate dimensions, we conclude that the charge-conjugation matrix must satisfy the relation

$$AD^*[\Lambda]A^{-1} = D[\Lambda]. \tag{2.5.89}$$

In terms of the generators of rotation, $\mathscr{J}_i$, and of boost, $\mathscr{K}_i$, with $i \in \{x, y, z\}$, of the representation $D[\Lambda]$, (2.5.89) can be expressed as

$$A\mathscr{J}_i^*A^{-1} = -\mathscr{J}_i \quad \text{and} \quad A\mathscr{K}_i^*A^{-1} = -\mathscr{K}_i. \tag{2.5.90}$$

Within the context of a given representation $D[\Lambda]$ of the restricted Lorentz group, the relations here derived allow for the charge-conjugation matrix to be determined up to a phase and an overall scale.

Before we move on to the next section, let us return to the constraints on the coefficient functions as given above in (2.5.81) and (2.5.82). As a special case of (2.5.89), in which $\Lambda = L(p)$, we obtain

$$AD^*[L(p)]A^{-1} = D[L(p)]. \tag{2.5.91}$$

Hence the constraints (2.5.81) and (2.5.82) are equivalent to the following at rest:

$$\sum_{\bar{l}} A_{l\bar{l}}u_{\bar{l}}^*(k; \sigma, s) = \frac{\lambda}{\kappa^*}\xi_s^{c*}\zeta_s^{c*}v_l(k; \sigma, s), \tag{2.5.92}$$

$$\sum_{\bar{l}} A_{l\bar{l}}v_{\bar{l}}^*(k; \sigma, s) = \frac{\kappa}{\lambda^*}\xi_s\zeta_s u_l(k; \sigma, s). \tag{2.5.93}$$

In Chs. 3 and 4 we will derive the charge-conjugation matrix via (2.5.90) and subsequently impose the constraints (2.5.92) and (2.5.93) upon the coefficient functions. A field $\psi(x)$ expanded in terms of coefficient functions that satisfy (2.5.92) and (2.5.93) will transform under charge-conjugation according to

$$C\psi_l(x)C^{-1} = \sum_{\bar{l}} A_{l\bar{l}}\psi_{\bar{l}}^*(x), \tag{2.5.94}$$

as desired.

In Ch. 3 and Ch. 4, we will derive the coefficient functions of the respective field theories in terms of a finite number of free parameters by first imposing the relations obtained in Sec. 2.5.3 from the transformation properties of the field under the unitary representations of the restricted Poincaré group. Subsequently, the constraints obtained above from the transformation properties of the field under the representations of the discrete symmetries are imposed in order to place further conditions on the remaining free parameters.

CP, CT, PT, and CPT

As one would expect, the succession of discrete symmetries does not yield any further constraints on the coefficient functions beyond those already given. We therefore summarise the results here as follows.

The succession CP, on the field, is given by

$$(CP)\,\psi(x)\,(CP)^{-1} = D[\mathscr{P}^{-1}]A\,\psi^*(\mathscr{P}x). \tag{2.5.95}$$

The succession CT, on the field, is given by

$$(CT)\,\psi(x)\,(CT)^{-1} = D[\mathscr{T}^{-1}]A\,\psi^*(\mathscr{T}x). \tag{2.5.96}$$

The succession PT, on the field, is given by

$$(PT)\,\psi(x)\,(PT)^{-1} = D[\mathscr{T}^{-1}]D[\mathscr{P}^{-1}]\,\psi(\mathscr{P}\mathscr{T}x). \tag{2.5.97}$$

The succession CPT, on the field, is given by

$$(CPT)\,\psi(x)\,(CPT)^{-1} = D[\mathscr{T}^{-1}]D[\mathscr{P}^{-1}]A\,\psi^*(\mathscr{P}\mathscr{T}x). \tag{2.5.98}$$

2.5.5 Constraints on coefficient functions: field (anti-) commutators

We briefly return to the (anti-) commutation relations of the field with itself and with its dual. As given in (2.5.15), these read

$$[\psi_l(x), \psi_{\bar{l}}(y)]_{\pm} = 0, \tag{2.5.99}$$

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_{\pm} = 0, \tag{2.5.100}$$

for $(x-y)^2 < 0$. We here provide expansions of these (anti-) commutators in terms of the coefficient functions so as to admit their direct application as constraints in the construction of field theories in Chs. 3 and 4.

It will be necessary here to consider two physically distinct scenarios. In the general case in which particles and antiparticles are distinct, it follows by direct substitution of $\psi(x)$ and $\bar{\psi}(x)$, as given in (2.5.13) and (2.5.16), that (2.5.99) vanishes identically on account of the (anti-) commutation relations of the creation and annihilation operators, as given in Sec. 2.4.

A non-trivial constraint on the coefficient functions is obtained from the second (anti-) commutator. Expanding the RHS of (2.5.100) we find

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_\pm = \sum_k \sum_{\sigma,s} \int \frac{d^3p}{(2\pi)^3} \eta_{\bar{l}k}(s) \Big[|\kappa|^2 \, u_l(p;\sigma,s) u_k^*(p;\sigma,s) \, \mathrm{e}^{-ip\cdot(x-y)} \\ \pm |\lambda|^2 \, v_l(p;\sigma,s) v_k^*(p;\sigma,s) \, \mathrm{e}^{+ip\cdot(x-y)} \Big] . \quad (2.5.101)$$

In the case of an indistinct antiparticle, we again obtain a non-trivial constraint from (2.5.100). It readily follows from the (anti-) commutation relations of the creation and annihilation operators, given in Sec. 2.4, that the expression of (2.5.100) in terms of the coefficient functions is identical to (2.5.101). A further constraint results from (2.5.99). Expanding the left hand side of (2.5.99), we obtain

$$[\psi_l(x), \psi_{\bar{l}}(y)]_\pm = \kappa\lambda \sum_{\sigma,s} \int \frac{d^3p}{(2\pi)^3} \Big[u_l(p;\sigma,s) v_{\bar{l}}(p;\sigma,s) \, \mathrm{e}^{-ip\cdot(x-y)} \\ \pm v_l(p;\sigma,s) u_{\bar{l}}(p;\sigma,s) \, \mathrm{e}^{+ip\cdot(x-y)} \Big] . \quad (2.5.102)$$

Before we move on to the next section, let us return to a remark made in Sec. 2.5, following the introduction of the above (anti-) commutation relations in (2.5.15), about the difference between the here considered $[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_\pm$, and the more familiar $[\psi_l(x), \psi_l^\dagger(y)]_\pm$. The difference is confined to the spin sums. From (2.5.101), we have the following two spin sums upon suppression of the column index:

$$\sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s)\boldsymbol{u}^\dagger(p;\sigma,s)\eta(s) \quad \text{and} \quad \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s)\boldsymbol{v}^\dagger(p;\sigma,s)\eta(s).$$

The corresponding spin sums obtained from the (anti-) commutator $[\psi_l(x), \psi_l^\dagger(y)]_\pm$ read

$$\sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s)\boldsymbol{u}^\dagger(p;\sigma,s) \quad \text{and} \quad \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s)\boldsymbol{v}^\dagger(p;\sigma,s).$$

It is thus apparent that replacing the Hermitian conjugate with the dual in (2.5.100) simply amounts to multiplication by the metric of the space of coefficient functions, the non-singular matrix η. If η is constant, this will have no bearing on the vanishing, or failure to vanish, of the RHS of (2.5.101). If, however, η has a non-trivial functional dependence on s, then the spin sums of $[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_\pm$ will differ from those of $[\psi_l(x), \psi_l^\dagger(y)]_\pm$. Although the latter is most prevalent in the literature [40–42, 82], it is in fact the former that must vanish as a necessary requirement in the Weinberg formalism for the Poincaré invariance of the S-matrix.

This completes our review and extension of the Weinberg formalism for the construction of quantum fields. In the next and final section of the current chapter we give a precise definition of the objective of the present work, that is, to explore an effect of lower spin components on the consistency and unitarity of massive bosonic quantum field theories of spin equal to or greater than one.

2.6 Consistency and unitarity

To put the present considerations into context, some remarks on higher spin quantum field theories will be in order. An important difference between higher spin quantum field theories and spin zero theories, such as that of a charged scalar field, is in the structure of their respective propagators. Unlike the propagator of a scalar field, the Feynman propagator, bosonic quantum field theories of spin equal to or greater than one have propagators the numerators of which contain powers of momentum equal to or greater than two, depending on the particular spin of the field under consideration. These powers of momentum give rise to divergences and loss of unitarity at high energies [101, 102] [103, Ch. 21]. A typical recipe to remove these offending elements is to introduce regulator terms. The following question thus arises: is it possible to construct massive bosonic theories of higher spin in such a way that the said offending terms in the propagator do not appear in the first place? We here expound our general approach and show by explicit example in Ch. 3 and Ch. 4 that the answer to the question at hand is in the affirmative.

Consider the propagator for a bosonic theory. As derived in App. B.4, this reads

$$\Delta(x-x') = \int \frac{d^4\rho}{(2\pi)^4}\,\mathrm{e}^{-i\rho\cdot(x-x')}\left[\frac{|\kappa|^2 N(\boldsymbol{p})\left(p^0+\rho^0\right)+|\lambda|^2 M(-\boldsymbol{p})\left(p^0-\rho^0\right)}{-\rho^2+m^2-i\epsilon}\right], \tag{2.6.1}$$

where $N(\boldsymbol{p})$ and $M(\boldsymbol{p})$ are the spin sums

$$N(\boldsymbol{p}) = \sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s)\boldsymbol{u}^\dagger(p;\sigma,s)\eta(s), \tag{2.6.2}$$

$$M(\boldsymbol{p}) = \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s)\boldsymbol{v}^\dagger(p;\sigma,s)\eta(s). \tag{2.6.3}$$

Here $\eta(s)$ is the metric, the matrix of the Lorentz invariant sesquilinear form defined on the coefficient functions as per the discussion in App. B.5. Given that p^0 and ρ^0 are independent parameters, the requirement that (2.6.1) be equal to the Feynman propagator (B.4.14), up to a multiplicative identity matrix, implies that the spin sums, (2.6.2) and (2.6.3), must satisfy

$$|\kappa|^2 N(\boldsymbol{p}) = |\lambda|^2 M(-\boldsymbol{p}) \quad \text{and} \quad |\kappa|^2 N(\boldsymbol{p}) = \frac{1}{2p^0}\mathbb{1}, \tag{2.6.4}$$

where $\mathbb{1}$ is an identity matrix of the appropriate dimensions.

Evaluating (2.6.4) at $p = k$, we obtain the following constraints on the coefficient functions at rest:

$$\sum_{\sigma,s} \boldsymbol{u}(k;\sigma,s)\boldsymbol{u}^\dagger(k;\sigma,s)\eta(s) = \frac{\mathbb{1}}{|\kappa|^2 2m}, \tag{2.6.5}$$

$$\sum_{\sigma,s} \boldsymbol{v}(k;\sigma,s)\boldsymbol{v}^\dagger(k;\sigma,s)\eta(s) = \frac{\mathbb{1}}{|\lambda|^2 2m}. \tag{2.6.6}$$

Rewriting the metric in terms of a spin-dependent function $\varpi(s)$ and a constant matrix β as $\eta(s) \equiv \varpi(s)\eta$ and using the identity $D[L(p)]\beta D^\dagger[L(p)] = \beta$, it is easy to show that conditions (2.6.5) and (2.6.6) are necessary and sufficient in order for (2.6.4) to be satisfied.

2.7 Conjecture on the significance of lower spin components

Let $\psi(x)$ be a quantum field that satisfies

$$U[\Lambda,a]\psi(x)U^{-1}[\Lambda,a] = D[\Lambda^{-1}]\psi(\Lambda x + a), \tag{2.7.1}$$

where $U[\Lambda,a]$ is a unitary representation of $\mathcal{P}_+^\uparrow$ and the matrix $D[\Lambda] \in (j/2, j/2), j \in \mathbb{Z}^*$, furnishes a representation of $\mathcal{L}_+^\uparrow$. There exists a dual field $\bar{\psi}(x)$ such that the vacuum expectation value of the time ordered product of $\bar{\psi}(y)\psi(x)$ will be proportional to the Feynman propagator if and only if $\psi(x)$ includes a sum over all $j+1$ spin degrees of freedom $s \in \{j, j-1, \ldots, 0\}$.

Proof. A sketch of what might later be extended to a proof:

- $\psi(x)$ will satisfy one or more field equations beyond the Klein-Gordon equation if and only if there are more field components than independent particle states [40, p. 200].

- The number of independent particle states will be equal to the number of field components if and only if $\psi(x)$ includes a sum over all $j+1$ spin degrees of freedom $s \in \{j, j-1, \ldots, 0\}$.

$\square$

3

Quantum field theory with spin one and spin zero degrees of freedom

The present chapter provides an application of the general formalism expounded in the preceding chapter. As a special case of the conjecture presented in Sec. 2.7 we here show that it is possible to construct a quantum field with spin one and spin zero degrees of freedom which, along with its dual, yields the Feynman spin one propagator. It is thus consistent and unitary at all energies without the need for regulator terms. We provide a derivation of the field theory including a free Lagrangian and Hamiltonian in terms of canonical field variables. The chapter concludes with remarks on some closely related phenomenological models that were extensively studied throughout the late 1960s and early to mid 1970s in the context of weak interactions.

3.1 The quantum field

The quantum field $\psi(x)$, as developed in Sec. 2.5, is given by the linear combination

$$\psi_l(x) = \sum_{\sigma,s} \int d^3p\,(2\pi)^{-3/2}\Big[\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p;\sigma,s)a(p;\sigma,s) \\ +\lambda\,\mathrm{e}^{+ip\cdot x}\,v_l(p;\sigma,s)b^\dagger(p;\sigma,s)\Big], \quad (3.1.1)$$

where $a(p;\sigma,s)$ is a particle annihilation operator and $b^\dagger(p;\sigma,s)$ is an antiparticle creation operator; $\kappa, \lambda \in \mathbb{C}$. The coefficient functions $\boldsymbol{u}(p;\sigma,s)$ and $\boldsymbol{v}(p;\sigma,s)$ are chosen such that $\psi(x)$ transforms according to

$$U[\Lambda,a]\,\psi_l(x)\,U^{-1}[\Lambda,a] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda^{-1}]\,\psi_{\bar{l}}(\Lambda x + a), \quad (3.1.2)$$

under the action of a unitary representation $U[\Lambda,a]$ of the restricted Poincaré group. The position independent matrix $D[\Lambda]$ furnishes a pseudounitary representation of the restricted Lorentz group on the space of coefficient functions; specifically, it is the $(1/2,1/2)$ representation, an irreducible 4×4 matrix representation given by the tensor product of the representations $(1/2,0)$ and $(0,1/2)$. The generators of the underlying Lie algebra are derived in App. C.7. The three generators of rotation read

$$\mathscr{J}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathscr{J}_y = \frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix},$$

$$\mathscr{J}_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.1.3}$$

The generators of Lorentz boost are

$$\mathscr{K}_x = \frac{1}{2}\begin{pmatrix} 0 & i & -i & 0 \\ i & 0 & 0 & -i \\ -i & 0 & 0 & i \\ 0 & -i & i & 0 \end{pmatrix}, \quad \mathscr{K}_y = \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix},$$

$$\mathscr{K}_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1.4}$$

The explicit form of the matrix representation $D[\Lambda]$ is obtained from the generators (3.1.3) and (3.1.4) via the exponential map, as shown in App. C.7.

3.2 Coefficient functions

We shall now derive the explicit form of the coefficient functions $\boldsymbol{u}(p;\sigma,s)$ and $\boldsymbol{v}(p;\sigma,s)$ using the constraints derived in the preceding chapter. The constraints that arise form the rotation subgroup of the Lorentz group are given in (2.5.58) to read

$$U\boldsymbol{J} = \boldsymbol{\mathscr{J}}U \quad \text{and} \quad -V\boldsymbol{J}^* = \boldsymbol{\mathscr{J}}V, \tag{3.2.1}$$

where $\boldsymbol{J}$, $\boldsymbol{\mathscr{J}}$, U and V are defined in (2.5.53)–(2.5.57). The explicit form of the components of $\boldsymbol{J} \equiv \boldsymbol{J}^{(0)} \oplus \boldsymbol{J}^{(1)}$ can be read off from (2.3.30). We thereby obtain

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1\sqrt{2} & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i/\sqrt{2} & 0 \\ 0 & i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 0 & i\sqrt{2} & 0 \end{pmatrix},$$

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.2.2}$$

The generators $\boldsymbol{\mathscr{J}}$, of the $(1/2, 1/2)$ representation, are given above in (3.1.3). Using these matrices in (3.2.1) and solving for U and V, we obtain the following coefficient functions. For $\boldsymbol{u}(k;\sigma,s)$, we have the following four coefficients: $\boldsymbol{u}(k;0,0)$, $\boldsymbol{u}(k;1,1)$, $\boldsymbol{u}(k;0,1)$, and $\boldsymbol{u}(k;-1,1)$. They read

$$\begin{pmatrix} 0 \\ -c_1 \\ c_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} c_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ c_2/\sqrt{2} \\ c_2/\sqrt{2} \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_2 \end{pmatrix}. \tag{3.2.3}$$

Similarly for $\boldsymbol{v}(k;\sigma,s)$, we again have four coefficients: $\boldsymbol{v}(k;0,0)$, $\boldsymbol{v}(k;1,1)$, $\boldsymbol{v}(k;0,1)$, and $\boldsymbol{v}(k;-1,1)$. They read

$$\begin{pmatrix} 0 \\ -c_3 \\ c_3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_4 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -c_4/\sqrt{2} \\ -c_4/\sqrt{2} \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} c_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.2.4}$$

Constants c_i, $i \in \{1,2,3,4\}$, are yet to be determined complex numbers. The validity of the labelling of the coefficient functions in terms of the eigenvalues of $\mathscr{J}_z$ and $\mathscr{J}^2 \equiv \mathscr{J}_x^2 + \mathscr{J}_y^2 + \mathscr{J}_z^2$ is easily checked. As expected, one finds

$$\mathscr{J}_z\, \boldsymbol{u}(k;\sigma,s) = \sigma\, \boldsymbol{u}(k;\sigma,s), \qquad \mathscr{J}^2 \boldsymbol{u}(k;\sigma,s) = s\,(s+1)\, \boldsymbol{u}(k;\sigma,s), \tag{3.2.5}$$

and

$$\mathscr{J}_z\, \boldsymbol{v}(k;\sigma,s) = -\sigma\, \boldsymbol{v}(k;\sigma,s), \qquad \mathscr{J}^2 \boldsymbol{v}(k;\sigma,s) = s\,(s+1)\, \boldsymbol{v}(k;\sigma,s). \tag{3.2.6}$$

Having thus computed the coefficient functions at rest, we know, from the results of Sec. 2.5.3, that they are given at momentum p^μ by

$$u_l(p;\sigma,s) = \sqrt{\frac{m}{p^0}} \sum_{\bar{l}} D_{l\bar{l}}[L(p)]\, u_{\bar{l}}(k;\sigma,s), \tag{3.2.7}$$

$$v_l(p;\sigma,s) = \sqrt{\frac{m}{p^0}} \sum_{\bar{l}} D_{l\bar{l}}[L(p)]\, v_{\bar{l}}(k;\sigma,s), \tag{3.2.8}$$

where $D[L(p)]$ is obtained via the exponential map from the generators (3.1.4), as per App. B.2.1.

3.3 The dual quantum field

Having defined the quantum field $\psi(x)$ and derived its expansion coefficients, we now turn to the dual field $\bar{\psi}(x)$. It is related to $\psi(x)$ by Hermitian conjugation and a conjugate-linear mapping of the coefficient functions, to be derived below. From the expansion of $\psi(x)$ given above in (3.1.1), we thus obtain

$$\bar{\psi}_l(x) = \sum_{\sigma,s} \int d^3p\, (2\pi)^{-3/2} \Big[\kappa^*\, \mathrm{e}^{+ip\cdot x}\, \bar{u}_l(p;\sigma,s) a^\dagger(p;\sigma,s) \\ + \lambda^*\, \mathrm{e}^{-ip\cdot x}\, \bar{v}_l(p;\sigma,s) b(p;\sigma,s)\Big], \tag{3.3.1}$$

where $a^\dagger(p;\sigma,s)$ and $b(p;\sigma,s)$ are particle creation and antiparticle annihilation operators, respectively; $\bar{u}_l(p;\sigma,s)$ and $\bar{v}_l(p;\sigma,s)$ are the dual coefficient functions, which we now derive. In doing so, we take the liberty of availing ourselves of the mathematical nomenclature of the dual without demanding all of the defining properties given in App. B.5. A few words on this matter shall thus be required.

It is clear by inspection that the set $\mathcal{S}_{\boldsymbol{u}}$, comprised of the coefficients $\boldsymbol{u}(k;\sigma,s)$ given in (3.2.3), constitutes a subspace, see Defn. 2, of a four-dimensional complex vector space. Furthermore, the elements of $\mathcal{S}_{\boldsymbol{u}}$ are linearly independent; thus, through an appropriately defined Hermitian form and suitably chosen values of c_1 and c_2, they can be made to yield an orthonormal basis. That is to say, we could invoke the mathematical machinery of App. B.5 to define a vector space and a corresponding dual space. We choose not to follow this approach because there appears to be no direct physical reason to demand that the coefficient functions must span a vector space or subspace.[1] Nevertheless, we require a well

[1] More specifically, there appears to to be no direct physical reason to demand that the coefficient functions must provide a basis for the representation $D[\Lambda]$. Nevertheless, the present work suggests that this condition may be necessary (though not sufficient) for the consistency and unitarity at high energies of bosonic

defined mapping from $\mathcal{S}_{\boldsymbol{u}}$ to $\mathbb{R}$. We do so via a sesquilinear form that is both Hermitian and invariant under the action of the Lorentz group. We thus also immediately obtain a mapping from the set $\mathcal{S}_{\boldsymbol{u}}$ to the set $\mathcal{S}_{\bar{\boldsymbol{u}}}$, a set of mappings from $\mathcal{S}_{\boldsymbol{u}}$ to $\mathbb{R}$. This set $\mathcal{S}_{\bar{\boldsymbol{u}}}$ itself is a vector subspace by Defn. 2. We will call $\mathcal{S}_{\bar{\boldsymbol{u}}}$ the dual space and use its elements, $\bar{\boldsymbol{u}}(k;\sigma,s)$ to construct $\bar{\psi}(x)$ as given above in Sec. 3.3.

We proceed as follows. Let $\mathcal{S}_{\boldsymbol{u}}$ be the non-empty set with elements given by (3.2.3) and let g be a sesquilinear form that maps the elements of $\mathcal{S}_{\boldsymbol{u}} \times \mathcal{S}_{\boldsymbol{u}}$ to the real line. Furthermore, let $\mathcal{B}$ be an ordered basis for $\mathcal{S}_{\boldsymbol{u}}$ and let η be the matrix of the sesquilinear form in that basis. We thus have from (B.5.1)

$$g(\boldsymbol{u},\boldsymbol{u}') = \boldsymbol{u}^\dagger \eta \boldsymbol{u}' \quad \forall \boldsymbol{u} \in \mathcal{S}_{\boldsymbol{u}}. \tag{3.3.2}$$

To constrain the sixteen components of η, we begin with the demand of G-invariance, that is, invariance under the action of an appropriate representation of $\mathcal{L}_+^\uparrow$. In the case at hand, this representation is defined locally by the generators (3.1.3) and (3.1.4). It is clear by inspection that these generators of rotation and boost are Hermitian and anti-Hermitian respectively. Hence, from (B.5.16), we have the conditions $\{\mathscr{K}_i,\eta\} = 0$ and $[\mathscr{J}_i,\eta] = 0$, for $i \in \{x,y,z\}$. Applying these to η, we obtain

$$\eta = \begin{pmatrix} -\varpi & 0 & 0 & 0 \\ 0 & 0 & -\varpi & 0 \\ 0 & -\varpi & 0 & 0 \\ 0 & 0 & 0 & -\varpi \end{pmatrix}, \tag{3.3.3}$$

where ϖ may, at most, depend on the labels s and σ. This is because η, by definition, is expressed in terms of the ordered basis $\mathcal{B}$ of $\mathcal{S}_{\boldsymbol{u}}$, a set the elements of which are labelled by a fixed value of momentum $p = k$ along with the pair of indices s and σ. We may thus rewrite (3.3.3) as

$$\eta(\sigma,s) = \varpi(\sigma,s)\beta, \tag{3.3.4}$$

where β is given by

$$\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.3.5}$$

Putting (3.3.2) and (3.3.4) together, we find the elements of the set $\mathcal{S}_{\bar{\boldsymbol{u}}}$ are given by

$$\mathcal{S}_{\bar{\boldsymbol{u}}} = \left\{ \bar{\boldsymbol{u}}(k;\sigma,s) = \boldsymbol{u}^\dagger(k;\sigma,s)\eta(\sigma,s) \mid \forall \boldsymbol{u}(k;\sigma,s) \in \mathcal{S}_{\boldsymbol{u}} \right\}. \tag{3.3.6}$$

Before we use the elements of $\mathcal{S}_{\bar{\boldsymbol{u}}}$ in the construction of $\bar{\psi}(x)$, we must first check that the coefficients, as given in (3.3.6), are consistent with the rotation constraints given above in Sec. 3.2. Some manipulation of (3.2.1) will be required.

We begin by noting that the map from $\boldsymbol{u}(k;\sigma,s)$ to $\bar{\boldsymbol{u}}(k;\sigma,s)$ is a one-to-one conju-

quantum field theories of spin equal to or greater than one.

gate linear mapping. Consequently, the rotation constraints on the elements of $\mathcal{S}_{\bar{\boldsymbol{u}}}$ may be derived from (3.2.1) via an appropriately defined transformation. Taking (3.2.1) and applying Hermitian conjugation on both sides, we obtain

$$\boldsymbol{J}^\dagger U^\dagger = U^\dagger \boldsymbol{\mathscr{J}}^\dagger. \tag{3.3.7}$$

Given that $\beta^2 = \mathbb{1}$ and $\boldsymbol{\mathscr{J}}^\dagger \beta = \beta \boldsymbol{\mathscr{J}}$, which follows from the Hermiticity of $\boldsymbol{\mathscr{J}}$ along with (3.3.4), we find that (3.3.7) is equivalent to

$$\boldsymbol{J}^\dagger U^\dagger \beta = U^\dagger \beta \boldsymbol{\mathscr{J}}. \tag{3.3.8}$$

Now, analogous to the definition of U given in (2.5.56), define $\overline{U}$ as

$$\overline{U} \equiv \left(\overline{U}_{l\alpha}\right) \equiv \left(\bar{u}_l(k;\sigma,s)\right), \tag{3.3.9}$$

where $\bar{\boldsymbol{u}}(k;\sigma,s)$ is as given in (3.3.6) and the index α takes the values $\{1,2,3,4\}$. Comparing $\overline{U}$ with $U^\dagger\beta$ we find that these differ in only one way: in contrast to $U^\dagger\beta$, each column in $\overline{U}$ has a further multiplicative factor $\varpi(\sigma,s)$. Looking back at (3.2.3) we see that, although the rotation constraint completely determines the relative scale and phase relationships within each spin-sector, no such constraint is placed on one spin-sector relative to another. The same is true here; the rotation constraints on $\bar{\boldsymbol{u}}(k;\sigma,s)$, as inferred from (3.3.7), allow ϖ to vary only with spin index s. The rotation constraint thus restricts the functional dependence of η, and thereby of ϖ, to s; hence, (3.3.4) becomes

$$\eta(s) = \varpi(s)\beta. \tag{3.3.10}$$

Accordingly, the dual space is now given by

$$\mathcal{S}_{\bar{\boldsymbol{u}}} = \left\{ \bar{\boldsymbol{u}}(k;\sigma,s) = \boldsymbol{u}^\dagger(k;\sigma,s)\eta(s) \mid \forall \boldsymbol{u}(k;\sigma,s) \in \mathcal{S}_{\boldsymbol{u}} \right\}. \tag{3.3.11}$$

Repeating the above for the $\boldsymbol{v}(k;\sigma,s)$ coefficients given in (3.2.4), we obtain the corresponding dual space:

$$\mathcal{S}_{\bar{\boldsymbol{v}}} = \left\{ \bar{\boldsymbol{v}}(k;\sigma,s) = \boldsymbol{v}^\dagger(k;\sigma,s)\eta(s) \mid \forall \boldsymbol{v}(k;\sigma,s) \in \mathcal{S}_{\boldsymbol{v}} \right\}. \tag{3.3.12}$$

The matrix β in the expression of $\eta(s)$ is identical for $\boldsymbol{u}(k;\sigma,s)$ and $\boldsymbol{v}(k;\sigma,s)$, given that $\boldsymbol{u}(k;\sigma,s)$ and $\boldsymbol{v}(k;\sigma,s)$ transform under the same representation of $\mathcal{L}_+^\uparrow$; one might, however, allow $\varpi(s)$ to be chosen independently. We do not consider this possibility here.

Having thus established that $\bar{\boldsymbol{u}}(k;\sigma,s)$ and $\bar{\boldsymbol{v}}(k;\sigma,s)$, as defined in (3.3.11) and (3.3.12), are consistent with the rotation constraints, we may now use these in the expansion of $\bar{\psi}(x)$, as per (3.3.1). The function $\varpi(s)$ as well as the four complex parameters in (3.2.3) and (3.2.4) are further constrained in the next section.

3.4 The propagator and the consistency and unitarity of quantum field theory

In order for the theory here under construction to be consistent and unitary at all energies without the need for regulator terms, the propagator must be equal to the Feynman propagator up to a multiplicative 4×4 identity matrix. According to the discussion of Sec. 2.6, this requirement yields the following constraint on the coefficient functions at rest:

$$\sum_{\sigma,s} \boldsymbol{u}(k;\sigma,s)\boldsymbol{u}^\dagger(k;\sigma,s)\varpi(s) = \frac{1}{|\kappa|^2 2m}\beta, \tag{3.4.1}$$

$$\sum_{\sigma,s} \boldsymbol{v}(k;\sigma,s)\boldsymbol{v}^\dagger(k;\sigma,s)\varpi(s) = \frac{1}{|\lambda|^2 2m}\beta. \tag{3.4.2}$$

Given that the overall scale of the coefficient functions is not yet fixed, we may choose without loss of generality $|\kappa| = |\lambda| = 1$. From (3.2.3), we compute the spin sums on the LHS of (3.4.1) to read

$$\begin{aligned}&\sum_{\sigma,s} \boldsymbol{u}(k;\sigma,s)\boldsymbol{u}^\dagger(k;\sigma,s)\varpi(s)\\ &= \begin{pmatrix} \varpi(1)|c_2|^2 & 0 & 0 & 0 \\ 0 & \varpi(1)\frac{|c_2|^2}{2} + \varpi(0)|c_1|^2 & \varpi(1)\frac{|c_2|^2}{2} - \varpi(0)|c_1|^2 & 0 \\ 0 & \varpi(1)\frac{|c_2|^2}{2} - \varpi(0)|c_1|^2 & \varpi(1)\frac{|c_2|^2}{2} + \varpi(0)|c_1|^2 & 0 \\ 0 & 0 & 0 & \varpi(1)|c_2|^2 \end{pmatrix}.\end{aligned}$$

Similarly for (3.4.2), we use (3.2.4) to obtain

$$\begin{aligned}&\sum_{\sigma,s} \boldsymbol{v}(k;\sigma,s)\boldsymbol{v}^\dagger(k;\sigma,s)\varpi(s)\\ &= \begin{pmatrix} \varpi(1)|c_4|^2 & 0 & 0 & 0 \\ 0 & \varpi(1)\frac{|c_4|^2}{2} + \varpi(0)|c_3|^2 & \varpi(1)\frac{|c_4|^2}{2} - \varpi(0)|c_3|^2 & 0 \\ 0 & \varpi(1)\frac{|c_4|^2}{2} - \varpi(0)|c_3|^2 & \varpi(1)\frac{|c_4|^2}{2} + \varpi(0)|c_3|^2 & 0 \\ 0 & 0 & 0 & \varpi(1)|c_4|^2 \end{pmatrix}.\end{aligned}$$

Hence, the constraints on the free parameters read

$$\varpi(1) = -1, \quad |c_2| = |c_4| = \frac{1}{\sqrt{2m}}, \quad \text{and} \quad \varpi(0)|c_1|^2 = \varpi(0)|c_3|^2 = \frac{1}{4m}. \tag{3.4.3}$$

We have uniquely determined the value of $\varpi(1)$. Although (3.4.3) does not fully constrain $\varpi(0)$, it does imply that it must be real and positive. We choose for simplicity $\varpi(0) = 1$. This has the advantage that $\eta(s)$ will be involutory not only for $s = 1$, but also for $s = 0$. The mapping to the dual spaces is thereby completely determined; consequently, the dual spaces $\mathcal{S}_{\bar{\boldsymbol{u}}}$ and $\mathcal{S}_{\bar{\boldsymbol{v}}}$ are determined uniquely in terms of the coefficient functions $\boldsymbol{u}(k;\sigma,s)$ and $\boldsymbol{v}(k;\sigma,s)$, respectively.

Inserting $\varpi(0) = 1$, the constraints (3.4.3) become

$$|c_1| = |c_3| = \frac{1}{\sqrt{4m}} \quad \text{and} \quad |c_2| = |c_4| = \frac{1}{\sqrt{2m}}. \tag{3.4.4}$$

Each of the four parameters is thus fixed up to a phase. For the further exploration of the coefficient functions, it will prove convenient to express the parameters c_i explicitly in terms of their magnitude and a phase. We thus write

$$c_1 = \frac{1}{\sqrt{4m}}\,\mathrm{e}^{i\phi_1}, \quad c_2 = \frac{1}{\sqrt{2m}}\,\mathrm{e}^{i\phi_2}, \quad c_3 = \frac{1}{\sqrt{4m}}\,\mathrm{e}^{i\phi_3}, \quad \text{and} \quad c_4 = \frac{1}{\sqrt{2m}}\,\mathrm{e}^{i\phi_4}\,. \tag{3.4.5}$$

Similarly for the constants κ and λ, we write

$$\kappa = \mathrm{e}^{i\phi_5} \quad \text{and} \quad \lambda = \mathrm{e}^{i\phi_6}\,. \tag{3.4.6}$$

In both (3.4.5) and (3.4.6), the implicitly defined parameters ϕ_i are yet unconstrained real numbers.

For future reference, the spin sums $N(\boldsymbol{p})$ and $M(\boldsymbol{p})$ are given by

$$N(\boldsymbol{p}) \equiv \sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s)\boldsymbol{u}^\dagger(p;\sigma,s)\beta\varpi(s) = \frac{1}{2p^0}\mathbb{1}_4, \tag{3.4.7}$$

$$M(\boldsymbol{p}) \equiv \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s)\boldsymbol{v}^\dagger(p;\sigma,s)\beta\varpi(s) = \frac{1}{2p^0}\mathbb{1}_4, \tag{3.4.8}$$

where $\mathbb{1}_4$ is a 4×4 identity matrix. The propagator (B.4.11) thus reads

$$\Delta_{F1}(x-x') = (2\pi)^{-4}\int d^4\rho\,\mathrm{e}^{-i\rho\cdot(x-x')}\left[\frac{\mathbb{1}_4}{-\rho^2+m^2-i\epsilon}\right], \tag{3.4.9}$$

where the subscript has been introduced to denote that this is the Feynman propagator for spin one; that is, the propagator $\Delta_F(x-y)$ given in (B.4.14) times the 4×4 identity matrix $\mathbb{1}_4$. The minus sign in front of the $\rho^2 \equiv \rho\cdot\rho$ term in the denominator, as well as that in the exponential, is a consequence of the metric convention as defined in (2.1.2).

3.5 Discrete symmetries

In order to further constrain the free parameters $\phi_1, \phi_2, \ldots, \phi_6$, we now impose the relations derived in Sec. 2.5.4 from the action of the representations of the discrete symmetries on $\psi(x)$. Unlike the treatment in the previous chapter, where the space-inversion and time-reversal matrices were denoted by $D[\mathscr{P}^{-1}]$ and $D[\mathscr{T}^{-1}]$, respectively, we here simply write $D[\mathscr{P}]$ and $D[\mathscr{T}]$. They are equivalent for the obvious reason that $\mathscr{P}$ and $\mathscr{T}$ are involutory.

3.5.1 Space-inversion

In Sec. 2.5.4, we derived the following constraints on the coefficient functions at rest:

$$D[\mathscr{P}]\boldsymbol{u}(k;\sigma,s) = \xi_s^* \boldsymbol{u}(k;\sigma,s), \tag{3.5.1}$$

$$D[\mathscr{P}]\boldsymbol{v}(k;\sigma,s) = \xi_s^c \boldsymbol{v}(k;\sigma,s). \tag{3.5.2}$$

Here $D[\mathscr{P}]$ is a 4×4 non-singular matrix, defined in (2.5.31) by the relations

$$D[\mathscr{P}]\mathscr{J}_i D^{-1}[\mathscr{P}] = +\mathscr{J}_i \quad \text{and} \quad D[\mathscr{P}]\mathscr{K}_i D^{-1}[\mathscr{P}] = -\mathscr{K}_i, \tag{3.5.3}$$

where $i \in \{x, y, z\}$. We immediately recognise (3.5.3) as the constraints, $\{\mathscr{K}_i, \eta\} = 0$ and $[\mathscr{J}_i, \eta] = 0$, used above in Sec. 3.3 to derive the matrix given in (3.3.3). Therefore,

$$D[\mathscr{P}] = \varrho\beta, \tag{3.5.4}$$

where $\varrho \in \mathbb{C}$. Applying this to the coefficient functions, we find

$$D[\mathscr{P}]\boldsymbol{u}(k;0,0) = \varrho\,\boldsymbol{u}(k;0,0), \qquad D[\mathscr{P}]\boldsymbol{u}(k;\sigma,1) = -\varrho\,\boldsymbol{u}(k;\sigma,1), \tag{3.5.5}$$

$$D[\mathscr{P}]\boldsymbol{v}(k;0,0) = \varrho\,\boldsymbol{v}(k;0,0), \qquad D[\mathscr{P}]\boldsymbol{v}(k;\sigma,1) = -\varrho\,\boldsymbol{v}(k;\sigma,1). \tag{3.5.6}$$

Thus, in order for the constraints (3.5.5) and (3.5.6) to be met, the following relation must be satisfied:

$$\xi_0^* = \xi_0^c = \varrho = -\xi_1^* = -\xi_1^c. \tag{3.5.7}$$

This shows that the complex number ϱ, in the parity matrix (3.5.4) above, must be of modulus one. Furthermore, the constraint (3.5.7) indicates that particles and antiparticles in the present theory must have the same intrinsic parity. On the other hand, the spin one sectors must be of opposite intrinsic parity as compared with the spin zero sector.

3.5.2 Time-reversal

The constraints on the coefficient functions at rest under the action of the time-reversal matrix are found in (2.5.71) and (2.5.72). Inserting κ and λ from (3.4.6) above, the constraints read

$$D[\mathscr{T}]\boldsymbol{u}(k;\sigma,s) = \mathrm{e}^{-2i\phi_5}\,\zeta_s^*(-)^{s+\sigma}\boldsymbol{u}^*(k;-\sigma,s), \tag{3.5.8}$$

$$D[\mathscr{T}]\boldsymbol{v}(k;\sigma,s) = \mathrm{e}^{-2i\phi_6}\,\zeta_s^c(-)^{s+\sigma}\boldsymbol{v}^*(k;-\sigma,s), \tag{3.5.9}$$

where, $D[\mathscr{T}]$ is a 4×4 non-singular matrix that satisfies the relations

$$D[\mathscr{T}]\mathscr{J}_i D^{-1}[\mathscr{T}] = -\mathscr{J}_i^* \quad \text{and} \quad D[\mathscr{T}]\mathscr{K}_i D^{-1}[\mathscr{T}] = +\mathscr{K}_i^*, \tag{3.5.10}$$

for $i\in\{x,y,z\}$. Explicit calculation using (3.1.3) and (3.1.4) yields

$$D[\mathscr{T}] = \begin{pmatrix} 0 & 0 & 0 & \varsigma \\ 0 & 0 & -\varsigma & 0 \\ 0 & -\varsigma & 0 & 0 \\ \varsigma & 0 & 0 & 0 \end{pmatrix}, \tag{3.5.11}$$

where $\varsigma\in\mathbb{C}$. Applying this to $\boldsymbol{u}(k;\sigma,s)$, we obtain

$$D[\mathscr{T}]\boldsymbol{u}(k;0,0) = \varsigma\,\mathrm{e}^{2i\phi_1}\,\boldsymbol{u}^*(k;0,0), \tag{3.5.12}$$

$$D[\mathscr{T}]\boldsymbol{u}(k;\sigma,1) = \varsigma\,\mathrm{e}^{2i\phi_2}(-)^{1+\sigma}\boldsymbol{u}^*(k;-\sigma,1). \tag{3.5.13}$$

Similarly for $\boldsymbol{v}(k;\sigma,s)$, we find

$$D[\mathscr{T}]\boldsymbol{v}(k;0,0) = \varsigma\,\mathrm{e}^{2i\phi_3}\,\boldsymbol{v}^*(k;0,0), \tag{3.5.14}$$

$$D[\mathscr{T}]\boldsymbol{v}(k;\sigma,1) = \varsigma\,\mathrm{e}^{2i\phi_4}(-)^{1+\sigma}\boldsymbol{v}^*(k;-\sigma,1). \tag{3.5.15}$$

Hence, in order for the constraints (3.5.8) and (3.5.9) to be satisfied, the phases ζ_s and ζ_s^c must be related to the free parameters $\phi_1,\phi_2,\ldots,\phi_6$ by

$$\zeta_0^* = \varsigma\,\mathrm{e}^{2i\phi_1+2i\phi_5}, \qquad \zeta_1^* = \varsigma\,\mathrm{e}^{2i\phi_2+2i\phi_5}, \tag{3.5.16}$$

$$\zeta_0^c = \varsigma\,\mathrm{e}^{2i\phi_3+2i\phi_6}, \qquad \zeta_1^c = \varsigma\,\mathrm{e}^{2i\phi_4+2i\phi_6}\,. \tag{3.5.17}$$

The complex number ς in the above time-reversal matrix must therefore be of modulus one.

3.5.3 Charge-conjugation

The constraints on the coefficient functions at rest under the action of the charge-conjugation matrix were derived in Sec. 2.5.4. From (2.5.92) and (2.5.93), with κ and λ as given in (3.4.6), we have

$$A\boldsymbol{u}^*(k;\sigma,s) = \mathrm{e}^{i\phi_5+i\phi_6}\,\xi_s^{c*}\zeta_s^{c*}\boldsymbol{v}(k;\sigma,s), \tag{3.5.18}$$

$$A\boldsymbol{v}^*(k;\sigma,s) = \mathrm{e}^{i\phi_5+i\phi_6}\,\xi_s\zeta_s\,\boldsymbol{u}(k;\sigma,s). \tag{3.5.19}$$

Inserting ξ and ζ from (3.5.7), (3.5.16), and (3.5.17), the constraints, for $s = 0$, become

$$A\boldsymbol{u}^*(k;0,0) = \mathrm{e}^{i\phi_5 - i\phi_6 - 2i\phi_3}\, \varrho^*\varsigma^*\boldsymbol{v}(k;0,0), \tag{3.5.20}$$

$$A\boldsymbol{v}^*(k;0,0) = \mathrm{e}^{-i\phi_5 + i\phi_6 - 2i\phi_1}\, \varrho^*\varsigma^*\boldsymbol{u}(k;0,0). \tag{3.5.21}$$

Similarly for $s = 1$, we have

$$A\boldsymbol{u}^*(k;\sigma,1) = -\,\mathrm{e}^{i\phi_5 - i\phi_6 - 2i\phi_4}\, \varrho^*\varsigma^*\boldsymbol{v}(k;\sigma,1), \tag{3.5.22}$$

$$A\boldsymbol{v}^*(k;\sigma,1) = -\,\mathrm{e}^{-i\phi_5 + i\phi_6 - 2i\phi_2}\, \varrho^*\varsigma^*\boldsymbol{u}(k;\sigma,1). \tag{3.5.23}$$

The charge-conjugation matrix A is non-singular, has dimensions 4×4, and is derived, according to (2.5.90), via the relations

$$A\mathscr{J}_i^*A^{-1} = -\mathscr{J}_i \quad \text{and} \quad A\mathscr{K}_i^*A^{-1} = -\mathscr{K}_i, \tag{3.5.24}$$

for $i \in \{x, y, z\}$. We thus obtain

$$A = \begin{pmatrix} 0 & 0 & 0 & -\delta \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ -\delta & 0 & 0 & 0 \end{pmatrix}, \tag{3.5.25}$$

with $\delta \in \mathbb{C}$. Applying this matrix to $\boldsymbol{u}(k;\sigma,s)$, we find

$$A\boldsymbol{u}^*(k;0,0) = +\delta\,\mathrm{e}^{-i\phi_1 - i\phi_3}\,\boldsymbol{v}(k;0,0), \tag{3.5.26}$$

$$A\boldsymbol{u}^*(k;\sigma,1) = -\delta\,\mathrm{e}^{-i\phi_2 - i\phi_4}\,\boldsymbol{v}(k;\sigma,1). \tag{3.5.27}$$

Likewise for $\boldsymbol{v}(k;\sigma,s)$, we have

$$A\boldsymbol{v}^*(k;0,0) = +\delta\,\mathrm{e}^{-i\phi_1 - i\phi_3}\,\boldsymbol{u}(k;0,0), \tag{3.5.28}$$

$$A\boldsymbol{v}^*(k;\sigma,1) = -\delta\,\mathrm{e}^{-i\phi_2 - i\phi_4}\,\boldsymbol{u}(k;\sigma,1). \tag{3.5.29}$$

Thus, in order for the constraints (3.5.20)–(3.5.23) to be met, the following relations must hold

$$\varrho\varsigma\delta = \mathrm{e}^{i\phi_1 - i\phi_3 + i\phi_5 - i\phi_6}, \tag{3.5.30}$$

$$\varrho\varsigma\delta = \mathrm{e}^{-i\phi_1 + i\phi_3 - i\phi_5 + i\phi_6}, \tag{3.5.31}$$

$$\varrho\varsigma\delta = \mathrm{e}^{i\phi_2 - i\phi_4 + i\phi_5 - i\phi_6}, \tag{3.5.32}$$

$$\varrho\varsigma\delta = \mathrm{e}^{-i\phi_2 + i\phi_4 - i\phi_5 + i\phi_6}. \tag{3.5.33}$$

As noted above, $|\varrho| = |\varsigma| = 1$; therefore, it must also be true that $|\delta| = 1$. Furthermore, taking (3.5.30) together with (3.5.31), it is clear that $\varrho\varsigma\delta = \pm 1$.

3.5.4 CPT

Before we move on to the next section, we briefly explore the implication of the above results upon the transformation property of the field under the succession of discrete symmetries CPT. From (2.5.98), we have

$$(CPT)\,\psi(x)\,(CPT)^{-1} = D[\mathscr{T}]D[\mathscr{P}]A\,\psi^*(\mathscr{P}\mathscr{T}x). \tag{3.5.34}$$

Now taking (3.5.4), (3.5.11), and (3.5.25), we compute the multiplicative matrix on the RHS of (3.5.34); we find $D[\mathscr{T}]D[\mathscr{P}]A = \varrho\varsigma\delta\mathbb{1}$. Hence, we obtain the following transformation property of the field under CPT:

$$(CPT)\,\psi(x)\,(CPT)^{-1} = \varrho\varsigma\delta\,\psi^*(-x), \tag{3.5.35}$$

where, by the constraints summarised in (3.5.30)–(3.5.33), the phase $\varrho\varsigma\delta$ is real and therefore may take on the values ±1, depending upon our choice of parameters $\phi_1, \phi_2, \ldots, \phi_6$. This result confirms the consistency of the above construct with the CPT theorem [40, Sec. 5.8].

3.6 Field commutators

Two further constraints that arise in the Weinberg formalism from the transformation properties of the S-matrix are those given in Sec. 2.5.5. For the bosonic fields at hand, these read

$$[\psi_l(x), \psi_{\bar{l}}(y)]_- = 0, \tag{3.6.1}$$

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_- = 0, \tag{3.6.2}$$

where x and y are space-like separated. We now consider each of these; first, in the case of a distinct antiparticle, then, in the case of an identical antiparticle.

In the case of a distinct antiparticle, we know from the discussion in Sec. 2.5.5 that (3.6.1) vanishes identically. This is not so for the second constraint (3.6.2). Computing the LHS of (3.6.2), as in (2.5.98), and recalling the spin sums $N(\boldsymbol{p})$ and $M(\boldsymbol{p})$, from (3.4.7) and (3.4.8) above, we have

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_- = \delta^l{}_{\bar{l}} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\left[\mathrm{e}^{-ip\cdot(x-y)} - \mathrm{e}^{+ip\cdot(x-y)}\right], \tag{3.6.3}$$

where $(\delta^l{}_{\bar{l}}) \equiv \mathbb{1}_4$. Looking at [40, p. 205], we find that (3.6.3) can be rewritten as

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_- = \delta^l{}_{\bar{l}}\,\Delta(x-y), \tag{3.6.4}$$

where

$$\Delta(x-y) \equiv \Delta_+(x-y) - \Delta_+(y-x), \tag{3.6.5}$$

and $\Delta_+(x-y)$ is the standard function

$$\Delta_+(x-y) = (2\pi)^{-3} \int \frac{d^3p}{2p^0}\, \mathrm{e}^{-ip\cdot(x-y)}, \tag{3.6.6}$$

with $p^0 \equiv \sqrt{|\boldsymbol{p}|^2+m^2}$. This is Lorentz invariant [40, p. 202] because $d^3p/\sqrt{|\boldsymbol{p}|^2+m^2}$ is the invariant volume element [40, p. 67]. It is easy to see that $\Delta_+(x-y)$ is even in a frame in which $x^0 = y^0$. By the Lorentz invariance of (3.6.6), we thus conclude that $\Delta_+(x-y)$ is even for any space-like interval. Consequently

$$\Delta(x-y) = 0, \quad \text{for} \quad (x-y)^2 < 0. \tag{3.6.7}$$

Therefore, (3.6.4) vanishes as desired for all space-like separated x and y.

In the case of an identical antiparticle, we have $a(p;\sigma,s) \equiv b(p;\sigma,s)$ and the LHS of (3.6.1) is expanded, as in (2.5.102), to read

$$\begin{aligned}[\psi_l(x), \psi_{\bar{l}}(y)]_- = \kappa\lambda \sum_{\sigma,s} \int \frac{d^3p}{(2\pi)^3} \Big[& u_l(p;\sigma,s) v_{\bar{l}}(p;\sigma,s)\, \mathrm{e}^{-ip\cdot(x-y)} \\ & - v_l(p;\sigma,s) u_{\bar{l}}(p;\sigma,s)\, \mathrm{e}^{+ip\cdot(x-y)} \Big]. \end{aligned} \tag{3.6.8}$$

Omitting the row index and computing the spin sums, using (3.2.3) and (3.2.4) along with (3.4.5), we find

$$\sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s)\boldsymbol{v}^T(p;\sigma,s) = \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s)\boldsymbol{u}^T(p;\sigma,s) \tag{3.6.9}$$

$$= \frac{-1}{2p^0m^2} S^\dagger \left([\chi_1 - \chi_2]\, p^\mu p^\nu + \chi_2 m^2 \eta^{\mu\nu} \right) S^*, \tag{3.6.10}$$

where $\chi_1 \equiv \mathrm{e}^{i(\phi_1+\phi_3)}$, $\chi_2 \equiv \mathrm{e}^{i(\phi_2+\phi_4)}$, and S is the constant unitary matrix

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & -i & 0 \\ -i & 0 & 0 & i \\ 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \end{pmatrix}. \tag{3.6.11}$$

There is no physical significance in this matrix; it is derived by the demand that it provide a similarity transformation from the matrix β to the Minkowski metric η. Application of this constraint to a 4×4 matrix with arbitrary complex components yields (3.6.11) up to multiplication by a complex number which is chosen here to be consistent with the literature [104, 105].

The object within the round brackets in (3.6.10) is a 4×4 matrix with components labelled by the spacetime the indices μ and ν. Substituting (3.6.10) into (3.6.8), the resulting expression can be written as

$$[\psi_l(x), \psi_{\bar{l}}(y)]_- = \kappa\lambda\, S^\dagger \left([\chi_1 - \chi_2] \frac{\partial^\mu \partial^\nu}{m^2} - \chi_2 \eta^{\mu\nu} \right) S^* \Delta(x-y), \tag{3.6.12}$$

and by (3.6.7) we have

$$[\psi_l(x), \psi_{\bar{l}}(y)]_- = 0, \quad \text{for} \quad (x-y)^2 < 0, \tag{3.6.13}$$

as required.

The evaluation of the second constraint, (3.6.2), is no different here in the case of an identical antiparticle as compared to the converse scenario above. It thus follows from (3.6.4) and (3.6.7) that (3.6.2) is satisfied.

3.7 Field commutators and discrete symmetries

As was discussed in Sec. 2.5.4, it follows as a consequence of the spin-dependence of the metric that the constraints imposed above on the coefficient functions under the action of the charge-conjugation matrix do not immediately guarantee that the charge-conjugation transformed field will commute with $\psi(x)$ and with $\bar{\psi}(x)$ at space-like separations. We must check two further conditions. From (2.5.87) and (2.5.88), we have the following:

$$\sum_{\sigma,s} \int \frac{d^3p}{(2\pi)^3} \Big[\boldsymbol{u}(p;\sigma,s) \boldsymbol{u}^\dagger(p;\sigma,s)\, \mathrm{e}^{-ip\cdot(x-y)} - \boldsymbol{v}(p;\sigma,s) \boldsymbol{v}^\dagger(p;\sigma,s)\, \mathrm{e}^{+ip\cdot(x-y)} \Big] = 0, \tag{3.7.1}$$

and (for an identical antiparticle)

$$\sum_{\sigma,s} \int \frac{d^3p}{(2\pi)^3} \Big[\boldsymbol{u}(p;\sigma,s) \boldsymbol{v}^T(p;\sigma,s) \eta(s)\, \mathrm{e}^{-ip\cdot(x-y)} - \boldsymbol{v}(p;\sigma,s) \boldsymbol{u}^T(p;\sigma,s) \eta(s)\, \mathrm{e}^{+ip\cdot(x-y)} \Big] = 0, \tag{3.7.2}$$

for all space-like separated x and y.

The constraint (3.7.1) is applicable for the case of a distinct as well as that of an indistinct antiparticle. Evaluating the spin sums, we obtain

$$\sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s) \boldsymbol{u}^\dagger(p;\sigma,s) = \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s) \boldsymbol{v}^\dagger(p;\sigma,s) = \frac{1}{2p^0 m^2} S^{-1} \left(2p^\mu p^\nu - m^2 \eta^{\mu\nu} \right) S.$$

Substituting into the LHS of (3.7.1), we may write the resulting expression in terms of $\Delta(x-y)$, such that

$$S^{-1}\left(-2\frac{\partial^\mu\partial^\nu}{m^2}-\eta^{\mu\nu}\right)S\Delta(x-y)=0, \quad \text{for} \quad (x-y)^2<0,$$

by (3.6.7). Thus, (3.7.1) is immediately satisfied without further constraints on the parameters $\phi_1, \phi_2, \ldots, \phi_6$.

As to (3.7.2), this constraint applies only to the case of an indistinct antiparticle. Computing the spin sums, we find

$$\begin{aligned}\sum_{\sigma,s}\boldsymbol{u}(p;\sigma,s)\bar{\boldsymbol{v}}^*(p;\sigma,s) &= \sum_{\sigma,s}\boldsymbol{v}(p;\sigma,s)\bar{\boldsymbol{u}}^*(p;\sigma,s)\\ &= \frac{-1}{2p^0m^2}S^\dagger\left([\chi_1+\chi_2]\,p^\mu p^\nu-\chi_2 m^2\eta^{\mu\nu}\right)S^*\beta,\end{aligned}$$

where χ_1 and χ_2 are as defined above following (3.6.10). Substituting into the LHS of (3.7.2), we obtain

$$S^\dagger\left([\chi_1+\chi_2]\,\frac{\partial^\mu\partial^\nu}{m^2}+\chi_2\eta^{\mu\nu}\right)S^*\beta\Delta(x-y)=0, \quad \text{for} \quad (x-y)^2<0, \tag{3.7.3}$$

by (3.6.7); hence, (3.7.2) is satisfied, again without further constraints on the parameters $\phi_1, \phi_2, \ldots, \phi_6$.

This completes the derivation of the quantum field $\psi(x)$ and that of the dual field $\bar{\psi}(x)$. There is still some remaining freedom in the choice of the parameters $\phi_1, \phi_2, \ldots, \phi_6$. In the final section of the present chapter, we formulate Lagrangian and Hamiltonian descriptions of the above field theory in terms of canonical field variables.

3.8 The canonical formalism

As shown by Weinberg in [40, p. 277], the Feynman propagator is the Green's function of the Klein-Gordon operator. The same is thus true of the propagator derived in the previous section. Applying the Klein-Gordon operator to (3.4.9), we obtain

$$\left(\partial^\mu\partial_\mu+m^2\right)\Delta_{F1}(x)=\mathbb{1}_4\int\frac{d^4\rho}{(2\pi)^4}\,\mathrm{e}^{-i\rho\cdot x}\left[\frac{-\rho^2+m^2}{-\rho^2+m^2-i\epsilon}\right]=\mathbb{1}_4\,\delta^4(x). \tag{3.8.1}$$

It follows that the free-field Lagrangian density is given, up to proportionality[2], by:

$$\mathscr{L}_0(x)=\bar{\psi}(x)\left(\overleftarrow{\partial}^\mu\overrightarrow{\partial}_\mu-m^2\right)\psi(x). \tag{3.8.2}$$

The subscript is to denote that the Lagrangian density is for a free-field theory, in agree-

[2] In the special case in which particles and antiparticles are identical, we must include an overall multiplicative factor of one half on the RHS of (3.8.2).

ment with the notation used for the free Hamiltonian in Sec. 2.5.

Accordingly, we propose the following canonically conjugate momenta:

$$\Pi(x) \equiv \frac{\partial \mathscr{L}_0(x)}{\partial(\partial_0 \psi(x))} = \partial^0 \bar{\psi}(x), \tag{3.8.3}$$

$$\bar{\Pi}(x) \equiv \frac{\partial \mathscr{L}_0(x)}{\partial(\partial_0 \bar{\psi}(x))} = \partial^0 \psi(x). \tag{3.8.4}$$

From (3.3.1), $\Pi(x)$ is given in terms of the creation and annihilation operators by

$$\Pi_l(x) = \sum_{\sigma,s} \int d^3p\, (2\pi)^{-3/2} \left(+ip^0\right) \Big[\kappa^* \,\mathrm{e}^{+ip\cdot x}\, \bar{u}_l(p;\sigma,s) a^\dagger(p;\sigma,s) \\ - \lambda^* \,\mathrm{e}^{-ip\cdot x}\, \bar{v}_l(p;\sigma,s) b(p;\sigma,s)\Big]\,. \tag{3.8.5}$$

Similarly from (3.1.1), $\bar{\Pi}(x)$ reads

$$\bar{\Pi}_l(x) = \sum_{\sigma,s} \int d^3p\, (2\pi)^{-3/2} \left(-ip^0\right) \big[\kappa \,\mathrm{e}^{-ip\cdot x}\, u_l(p;\sigma,s) a(p;\sigma,s) \\ - \lambda \,\mathrm{e}^{+ip\cdot x}\, v_l(p;\sigma,s) b^\dagger(p;\sigma,s)\Big]\,. \tag{3.8.6}$$

In the next section we check the consistency of the above identifications by computing the canonical commutation relations [40, p. 293].

3.8.1 Locality

If $\psi(x)$, $\bar{\psi}(x)$, $\Pi(x)$ and $\bar{\Pi}(x)$ are to provide a consistent interpretation as independent canonical field variables, then the following equal time commutators must be satisfied:

$$[\psi_l(\boldsymbol{x},t), \psi_{\bar{l}}(\boldsymbol{y},t)]_- = [\bar{\psi}_l(\boldsymbol{x},t), \bar{\psi}_{\bar{l}}(\boldsymbol{y},t)]_- = [\psi_l(\boldsymbol{x},t), \bar{\psi}_{\bar{l}}(\boldsymbol{y},t)]_- = 0, \tag{3.8.7}$$

$$[\Pi_l(\boldsymbol{x},t), \Pi_{\bar{l}}(\boldsymbol{y},t)]_- = [\bar{\Pi}_l(\boldsymbol{x},t), \bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = [\Pi_l(\boldsymbol{x},t), \bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = 0, \tag{3.8.8}$$

$$[\psi_l(\boldsymbol{x},t), \bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = [\bar{\psi}_l(\boldsymbol{x},t), \Pi_{\bar{l}}(\boldsymbol{y},t)]_- = 0. \tag{3.8.9}$$

Further, in order that $\psi(x)$ and $\Pi(x)$ be conjugate, and likewise for $\bar{\psi}(x)$ and $\bar{\Pi}(x)$, it is required that

$$[\psi_l(\boldsymbol{x},t), \Pi_{\bar{l}}(\boldsymbol{y},t)]_- = [\bar{\psi}_l(\boldsymbol{x},t), \bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = i\, \delta^l{}_{\bar{l}}\, \delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{3.8.10}$$

We now evaluate each of these for the two physically distinct scenarios at hand, beginning with the case of a distinct antiparticle.

For a distinct antiparticle

Here it follows from the commutation relations of the creation and annihilation operators given in (2.4.3) and (2.4.4) that the first two commutators in (3.8.7) and likewise the first two commutators in (3.8.8) and the two commutators in (3.8.9) trivially vanish. Furthermore, the last commutator in (3.8.7) vanishes as a special case of (3.6.4). As for the last commutator in (3.8.8), this is expanded and evaluated using the transpose of the spin sums given in (3.4.7) and (3.4.8). We thus obtain

$$[\Pi_l(\boldsymbol{x},t),\bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}p^0p^0\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right]. \tag{3.8.11}$$

This clearly vanishes as required. We are now left only with (3.8.10). Here an explicit calculation is required.

The commutator of the field with its prospective canonically conjugate momentum is expanded using the bosonic commutators of the creation and annihilation operators to read

$$\begin{aligned}[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\sum_{\sigma,s}\int \frac{d^3p}{(2\pi)^3}p^0\Big[&u_l(p;\sigma,s)\bar{u}_{\bar{l}}(p;\sigma,s)\,\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\\ &+v_l(p;\sigma,s)\bar{v}_{\bar{l}}(p;\sigma,s)\,\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\Big].\end{aligned} \tag{3.8.12}$$

Substituting the spin sums (3.4.7) and (3.4.8), this becomes

$$[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + \mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right].$$

A change of variables then yields

$$[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\int \frac{d^3p}{(2\pi)^3}\,\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}.$$

Identifying the three-dimensional integral representation of the Dirac delta function, we obtain

$$[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{3.8.13}$$

in accordance with (3.8.10).

The final commutator is that of $\bar{\psi}(x)$ with $\bar{\Pi}(y)$ at equal time. Expanding this in a similar manner as in the case of the preceding commutator, we obtain

$$\begin{aligned}[\bar{\psi}_l(\boldsymbol{x},t),\bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = i\sum_{\sigma,s}\int \frac{d^3p}{(2\pi)^3}p^0\Big[&\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\bar{u}_l(p;\sigma,s)u_{\bar{l}}(p;\sigma,s)\\ &+\mathrm{e}^{+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\bar{v}_l(p;\sigma,s)v_{\bar{l}}(p;\sigma,s)\Big].\end{aligned} \tag{3.8.14}$$

Of course, $\bar{u}_l(p;\sigma,s)u_{\bar{l}}(p;\sigma,s) = u_{\bar{l}}(p;\sigma,s)\bar{u}_l(p;\sigma,s)$ and likewise for the components of the $\boldsymbol{v}$ coefficient functions; hence, by the previous calculation of the commutator of the field with its canonically conjugate momentum, (3.8.14) is simply

$$[\bar{\psi}_l(\boldsymbol{x},t),\bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{3.8.15}$$

as required by (3.8.10). This completes the analysis of the locality properties of the above field operators in the case of a distinct antiparticle. We now turn to the alternate scenario.

For an indistinct antiparticle

Unlike the previous case, in which the commutation relations of the fields under consideration were in many cases determined completely by the commutation relations of the creation and annihilation operators alone, it will be necessary here to explore the explicit properties of the relevant spin sums. For this reason it will prove most convenient to rewrite the coefficient functions in the vector basis; that is, the basis in which the row index l, which labels the components of the expansion coefficients, is replaced by a spacetime index μ. Recalling the matrix S given above in (3.6.11) we note that $SD[\Lambda]S^{-1} = (\Lambda^\mu{}_\nu)$. Therefore, we can write the coefficient functions at rest in terms of the new basis as

$$(u^\mu(k;\sigma,s)) = S\boldsymbol{u}(k;\sigma,s) \quad \text{and} \quad (v^\mu(k;\sigma,s)) = S\boldsymbol{v}(k;\sigma,s). \tag{3.8.16}$$

These are related to the coefficient functions at arbitrary momentum via

$$u^\mu(p;\sigma,s) = L(p)^\mu{}_\nu u^\nu(k;\sigma,s) \quad \text{and} \quad v^\mu(p;\sigma,s) = L(p)^\mu{}_\nu v^\nu(k;\sigma,s), \tag{3.8.17}$$

where $\mu \in \{0,1,2,3\}$. The field thus reads

$$\psi^\mu(x) = \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\sqrt{\frac{m}{p^0}}\,\Big[\kappa\,\mathrm{e}^{-ip\cdot x}\,u^\mu(p;\sigma,s)a(p;\sigma,s) + \lambda\,\mathrm{e}^{+ip\cdot x}\,v^\mu(p;\sigma,s)a^\dagger(p;\sigma,s)\Big]. \tag{3.8.18}$$

Similarly the dual field becomes

$$\psi^*_\mu(x) = \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\sqrt{\frac{m}{p^0}}\,\Big[\kappa^*\,\mathrm{e}^{+ip\cdot x}\,u^*_\mu(p;\sigma,s)a^\dagger(p;\sigma,s) + \lambda^*\,\mathrm{e}^{-ip\cdot x}\,v^*_\mu(p;\sigma,s)a(p;\sigma,s)\Big], \tag{3.8.19}$$

where

$$u_\mu(p;\sigma,s) \equiv \eta_{\mu\nu}(s)u^\nu(p;\sigma,s) \quad \text{and} \quad v_\mu(p;\sigma,s) \equiv \eta_{\mu\nu}(s)v^\nu(p;\sigma,s), \tag{3.8.20}$$

with $(\eta_{\mu\nu}(s)) = S\eta(s)S^{-1}$. Furthermore, the fields $\Pi^\mu(x) \equiv \partial_0\psi^*_\mu(x)$ and $\Pi^*_\mu(x) \equiv \partial_0\psi^\mu(x)$, respectively, by (3.8.19) and (3.8.18), are expanded to read

$$\Pi^\mu(x) = i\sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\sqrt{mp^0}\Big[\kappa^*\,\mathrm{e}^{+ip\cdot x}\,u^*_\mu(p;\sigma,s)a^\dagger(p;\sigma,s) \\ -\lambda^*\,\mathrm{e}^{-ip\cdot x}\,v^*_\mu(p;\sigma,s)a(p;\sigma,s)\Big]\,, \quad (3.8.21)$$

and

$$\Pi^*_\mu(x) = -i\sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\sqrt{mp^0}\Big[\kappa\,\mathrm{e}^{-ip\cdot x}\,u^\mu(p;\sigma,s)a(p;\sigma,s) \\ -\lambda\,\mathrm{e}^{+ip\cdot x}\,v^\mu(p;\sigma,s)a^\dagger(p;\sigma,s)\Big]\,. \quad (3.8.22)$$

In order to determine whether these may serve as canonical field variables, we again compute the equal time commutators (3.8.7)–(3.8.10).

Given that the number of degrees of freedom of the system has decreased by a factor of one half, on account of the identification $a(p;\sigma,s) \equiv b(p;\sigma,s)$, we expect that the above proposed canonical field variables will fail to be independent. To confirm this, it is sufficient to compute the two commutators given above in (3.8.9). For the first of these we obtain

$$[\psi^\mu(\boldsymbol{x},t),\Pi^*_\nu(\boldsymbol{y},t)]_- \\ = -i\frac{\kappa\lambda}{m^2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}+\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right]\left([\chi_1-\chi_2]\,p^\mu p^\nu+\chi_2\eta^{\mu\nu}m^2\right)\,, \quad (3.8.23)$$

where we have made use of the spin sums

$$\sum_{\sigma,s}u^\mu(p;\sigma,s)v^\nu(p;\sigma,s) = \sum_{\sigma,s}v^\mu(p;\sigma,s)u^\nu(p;\sigma,s) \\ = \left([\chi_1-\chi_2]\,p^\mu p^\nu+\chi_2\eta^{\mu\nu}m^2\right)\,. \quad (3.8.24)$$

Similarly, the second commutator in (3.8.9) is expanded to read

$$\left[\psi^*_\mu(\boldsymbol{x},t),\Pi^\nu(\boldsymbol{y},t)\right]_- \\ = i\frac{\kappa^*\lambda^*}{m^2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}+\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right]\left([\chi^*_1-\chi^*_2]\,p_\mu p_\nu+\chi^*_2\eta_{\mu\nu}m^2\right)\,. \quad (3.8.25)$$

Clearly these are non-vanishing. Thus by (3.8.23) $\psi^\mu(\boldsymbol{x},t)$ and $\Pi^*_\nu(\boldsymbol{y},t)$ fail to be independent. Likewise by (3.8.25) and $\psi^*_\mu(\boldsymbol{x},t)$ and $\Pi^\nu(\boldsymbol{y},t)$ fail to be independent, as expected. We therefore confine to $\psi^\mu(x)$ and $\Pi^\mu(x)$ for prospective canonical field variables. In order for these to yield a consistent interpretation, they must satisfy the following commutation relations:

$$[\psi^\mu(\boldsymbol{x},t),\psi^\nu(\boldsymbol{y},t)]_- = 0, \tag{3.8.26}$$

$$[\Pi^\mu(\boldsymbol{x},t),\Pi^\nu(\boldsymbol{y},t)]_- = 0, \tag{3.8.27}$$

$$[\psi^\mu(\boldsymbol{x},t),\Pi^\nu(\boldsymbol{y},t)]_- = i\,\delta^\mu{}_\nu\,\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{3.8.28}$$

Computing each of these in turn, we obtain for (3.8.26)

$$\begin{aligned}&[\psi^\mu(\boldsymbol{x},t),\psi^\nu(\boldsymbol{y},t)]_-\\ &\quad= -\frac{\kappa\lambda}{m^2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\left([\chi_1-\chi_2]\,p^\mu p^\nu+\chi_2 m^2\eta^{\mu\nu}\right)\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}-\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right].\end{aligned}$$

Similarly for (3.8.27), we have

$$\begin{aligned}&[\Pi^\mu(\boldsymbol{x},t),\Pi^\nu(\boldsymbol{y},t)]_-\\ &\quad= -\frac{\kappa^*\lambda^*}{m^2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}-\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right]\left([\chi_1^*-\chi_2^*]\,p_\mu p_\nu+\chi_2^*\eta_{\mu\nu}m^2\right).\end{aligned} \tag{3.8.29}$$

Lastly for (3.8.28), we find

$$[\psi^\mu(\boldsymbol{x},t),\Pi^\nu(\boldsymbol{y},t)]_- = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}+\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right] \tag{3.8.30}$$

$$= i\,\delta^\mu{}_\nu\,\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{3.8.31}$$

It is thus apparent that $\psi^\mu(x)$ and $\Pi^\mu(x)$ will satisfy the equal time commutators (3.8.26)–(3.8.28) if we choose the phases such that $\chi_1=\chi_2$, that is, by the definition of χ_1 and χ_2 given above in the text following (3.6.10), we require

$$\mathrm{e}^{i(\phi_1+\phi_3)} = \mathrm{e}^{i(\phi_2+\phi_4)}. \tag{3.8.32}$$

This choice of phases is allowable, given the remaining freedom as per (3.5.30)–(3.5.33).

Having thus found a consistent set of canonical field variables, first for the case of a distinct, and then for that of an indistinct antiparticle, we proceed to develop the Hamiltonian formalism. We give explicit consideration only the case of an distinct antiparticle. The alternate scenario follows trivially.

3.8.2 The Hamiltonian

From the above free-field Lagrangian and canonical field variables we obtain the following expression for $\mathcal{H}_0$:

$$\mathcal{H}_0 = \int d^3x\,\mathscr{H}_0(x), \tag{3.8.33}$$

where

$$\mathscr{H}_0(x) = \Pi(x)\partial_0\psi(x) + \bar{\Pi}(x)\partial_0\bar{\psi}(x) - \mathscr{L}_0(x). \tag{3.8.34}$$

We now verify that (3.8.33) reduces to the free-particle Hamiltonian upon explicit evaluation in terms of the creation and annihilation operators.

Inserting (3.8.3), (3.8.4), and (3.8.2), we obtain

$$\begin{aligned}\mathscr{H}_0(x) &= \partial^0\bar{\psi}(x)\partial_0\psi(x) + \partial^0\psi(x)\partial_0\bar{\psi}(x) - \partial^\mu\bar{\psi}(x)\partial_\mu\psi(x) + m^2\bar{\psi}(x)\psi(x)\\ &= \partial^0\psi(x)\partial_0\bar{\psi}(x) + \nabla\bar{\psi}(x)\cdot\nabla\psi(x) + m^2\bar{\psi}(x)\psi(x).\end{aligned} \tag{3.8.35}$$

For the sake of notational convenience we suppress the energy component in the coefficient functions and creation and annihilation operators: instead of writing $\boldsymbol{u}(p;\sigma,s)$, for instance, we simply write $\boldsymbol{u}(\boldsymbol{p};\sigma,s)$. Given that the momenta are assumed to be on-shell these expressions are functionally equivalent. Substituting (3.8.35) into (3.8.33) we begin by evaluating the first term. This yields

$$\begin{aligned}\sum_{\sigma,s}\sum_{\sigma',s'}\int d^3p\,(p^0p^0)\Big[&|\kappa|^2\bar{\boldsymbol{u}}(\boldsymbol{p};\sigma',s')\boldsymbol{u}(\boldsymbol{p};\sigma,s)a(\boldsymbol{p};\sigma,s)a^\dagger(\boldsymbol{p};\sigma',s')\\ &+|\lambda|^2\bar{\boldsymbol{v}}(\boldsymbol{p};\sigma',s')\boldsymbol{v}(\boldsymbol{p};\sigma,s)b^\dagger(\boldsymbol{p};\sigma,s)b(\boldsymbol{p};\sigma',s')\\ &-\kappa^*\lambda\,\mathrm{e}^{+2ip^0t}\,\bar{\boldsymbol{u}}(-\boldsymbol{p};\sigma',s')\boldsymbol{v}(\boldsymbol{p};\sigma,s)b^\dagger(\boldsymbol{p};\sigma,s)a^\dagger(-\boldsymbol{p};\sigma',s')\\ &-\kappa\lambda^*\,\mathrm{e}^{-2ip^0t}\,\bar{\boldsymbol{v}}(-\boldsymbol{p};\sigma',s')\boldsymbol{u}(\boldsymbol{p};\sigma,s)a(\boldsymbol{p};\sigma,s)b(-\boldsymbol{p};\sigma',s')\Big].\end{aligned}$$

For the remaining two terms we obtain

$$\begin{aligned}\sum_{\sigma,s}\sum_{\sigma',s'}\int d^3p\,(|\boldsymbol{p}|^2+m^2)\Big[&|\kappa|^2\bar{\boldsymbol{u}}(\boldsymbol{p};\sigma,s)\boldsymbol{u}(\boldsymbol{p};\sigma',s')a^\dagger(\boldsymbol{p};\sigma,s)a(\boldsymbol{p};\sigma',s')\\ &+|\lambda|^2\bar{\boldsymbol{v}}(\boldsymbol{p};\sigma,s)\boldsymbol{v}(\boldsymbol{p};\sigma',s')b(\boldsymbol{p};\sigma,s)b^\dagger(\boldsymbol{p};\sigma',s')\\ &+\kappa^*\lambda\,\mathrm{e}^{+2ip^0t}\,\bar{\boldsymbol{u}}(\boldsymbol{p};\sigma,s)\boldsymbol{v}(-\boldsymbol{p};\sigma',s')a^\dagger(\boldsymbol{p};\sigma,s)b^\dagger(-\boldsymbol{p};\sigma',s')\\ &+\kappa\lambda^*\,\mathrm{e}^{-2ip^0t}\,\bar{\boldsymbol{v}}(\boldsymbol{p};\sigma,s)\boldsymbol{u}(-\boldsymbol{p};\sigma',s')b(\boldsymbol{p};\sigma,s)a(-\boldsymbol{p};\sigma',s')\Big].\end{aligned}$$

These expressions can be further simplified using the following orthogonality relations:

$$\bar{\boldsymbol{u}}(\boldsymbol{p};\sigma,s)\boldsymbol{u}(\boldsymbol{p};\sigma',s') = \frac{1}{2p^0}\delta_{\sigma\sigma'}\delta_{ss'}, \tag{3.8.36}$$

$$\bar{\boldsymbol{v}}(\boldsymbol{p};\sigma,s)\boldsymbol{v}(\boldsymbol{p};\sigma',s') = \frac{1}{2p^0}\delta_{\sigma\sigma'}\delta_{ss'}. \tag{3.8.37}$$

Recalling $\kappa = \mathrm{e}^{i\phi_5}$ and $\lambda = \mathrm{e}^{i\phi_6}$ as per (3.4.6), we may thus express $\mathcal{H}_0$ as

$$\mathcal{H}_0 = \frac{1}{2}\sum_{\sigma,s}\int d^3p\, p^0 \Big[a^\dagger(\boldsymbol{p};\sigma,s)a(\boldsymbol{p};\sigma,s) + b(\boldsymbol{p};\sigma,s)b^\dagger(\boldsymbol{p};\sigma,s) + a(\boldsymbol{p};\sigma,s)a^\dagger(\boldsymbol{p};\sigma,s) + b^\dagger(\boldsymbol{p};\sigma,s)b(\boldsymbol{p};\sigma,s)\Big]. \tag{3.8.38}$$

We now avail ourselves of the commutation relations of the creation and annihilation operators to write (3.8.38) in normal ordered form. From (2.4.3) and (2.4.4), we thus obtain

$$\mathcal{H}_0 = \sum_{\sigma,s}\int d^3p\, p^0 \Big[a^\dagger(\boldsymbol{p};\sigma,s)a(\boldsymbol{p};\sigma,s) + b^\dagger(\boldsymbol{p};\sigma,s)b(\boldsymbol{p};\sigma,s) + \delta(\boldsymbol{p}-\boldsymbol{p})\Big], \tag{3.8.39}$$

in agreement with the result given by Weinberg in [40, p. 296] for the free-particle Hamiltonian.

3.9 Phenomenological models

The idea to include one or more spin zero degrees of freedom in order to achieve renormalisability has a history in the context of weak interactions. In the late 1960s and early 1970s, it was proposed by Lee, Wick, and Gell-Mann *et al.* [106–109] that weak interactions might be mediated by one or more intermediate scalar bosons along with an intermediate vector boson. The theoretical motivation was based upon the observation that divergences could be ameliorated if one included a spin zero component in the propagator. This spin zero component was introduced with a minus sign by means of an indefinite metric in the space of physical states, a possibility that arose in the 1940s from the work of Dirac [110] and Pauli [111, 112]. Detailed phenomenological analyses were conducted by several authors [109, 113–117] in the 1970s.

4

Quantum field theory with spin two, spin one, and spin zero degrees of freedom

The present chapter constitutes the second example of the general multi-spin formalism developed in Ch. 2. We begin by constructing a field theory that includes spin two, spin one, and spin zero degrees of freedom. The vacuum expectation value of the relevant quantum field and its dual yields the Feynman spin two propagator. The theory is thus consistent and unitary at all energies without the need for regulator terms. The penultimate section provides the associated free-field Lagrangian in terms of canonical field variables. The resultant Hamiltonian is evaluated in terms of creation and annihilation operators and is found to be consistent with expectation: the free-particle Hamiltonian plus vacuum energy.

4.1 The quantum field

As in the example of Ch. 3, the quantum field $\psi(x)$ takes the form

$$\psi_l(x) = \sum_{\sigma,s} \int d^3p\,(2\pi)^{-3/2}\,\Big[\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p;\sigma,s)a(p;\sigma,s) + \lambda\,\mathrm{e}^{+ip\cdot x}\,v_l(p;\sigma,s)b^\dagger(p;\sigma,s)\Big]\,, \quad (4.1.1)$$

where $a(p;\sigma,s)$ is a particle annihilation operator and $b^\dagger(p;\sigma,s)$ is an antiparticle creation operator, κ and λ are complex valued constants, and the coefficient functions $\boldsymbol{u}(p;\sigma,s)$ and $\boldsymbol{v}(p;\sigma,s)$ are chosen such that $\psi(x)$ transforms according to

$$U[\Lambda,a]\,\psi_l(x)\,U^{-1}[\Lambda,a] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda^{-1}]\,\psi_{\bar{l}}(\Lambda x + a), \quad (4.1.2)$$

under the action of a unitary representation $U[\Lambda,a]$ of the restricted Poincaré group. The position independent matrix $D[\Lambda]$ furnishes a pseudounitary representation of the Lorentz group on the space of coefficient functions, provided these span a vector space of the appropriate dimensions. We here choose $D[\Lambda]$ to be $(1,1)$, an irreducible 9×9 matrix representation given by the tensor product: $(1,1) \equiv (1,0) \otimes (0,1)$. Before we can begin to constraint the coefficient functions in (4.1.1), we must first derive the generators of rotation and boost that define the $(1,1)$ representation.

In App. C.5, we find two isomorphic spin one representations of the Lie algebra of the rotation group: the standard representation given by (2.3.30) with $s=1$, and the so called adjoint representation. We shall have more to say about the adjoint representation in Ch. 7. For now it will suffice to note that the choice of the adjoint representation in the present context is computationally convenient, as will be demonstrated in Sec. 4.2. The rotation generators of the adjoint representation read

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.1.3}$$

For the representation $(1,0)$, the corresponding boost generators are given by $\mathcal{K}_k = -i\mathcal{J}_k$, with $k \in \{x,y,z\}$. The representation $(0,1)$ is defined by the same rotation generators; however, the generators of boost are given by $\mathcal{K}_k = i\mathcal{J}_k$. As derived in App. C.5.1, the rotation operator for $(1,0)$ is given by

$$R^{(1,0)} \equiv \exp[i\boldsymbol{J}\cdot\boldsymbol{\theta}] = \mathbb{1}_3 + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\,[\sin(\theta)] + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2\,[1-\cos(\theta)]\,, \tag{4.1.4}$$

where $\hat{\boldsymbol{\theta}}$ defines the axis of rotation and θ is the angle of rotation about that axis. Given that the rotation generators of $(1,0)$ are identical to those of $(0,1)$, the same is true of the rotation operators; that is, $R^{(1,0)} \equiv R^{(0,1)}$. The rotation operator of the $(1,0)\otimes(0,1)$ representation, $D[\mathscr{R}] \equiv R^{(1,1)}$, is thus given by

$$D[\mathscr{R}] \equiv R^{(1,0)} \otimes R^{(0,1)}. \tag{4.1.5}$$

The underlying generators are computed via the usual technique:

$$\mathscr{J}_k = \frac{1}{i}\frac{\partial D[\mathscr{R}]}{\partial\theta_k}\bigg|_{\theta_k\to 0}. \tag{4.1.6}$$

The rotation generators $\mathscr{J}_x$, $\mathscr{J}_y$, and $\mathscr{J}_z$, respectively, are thus found to read

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & i & 0 & i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & i & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -i & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1.7}$$

Now, for the corresponding generators of boost, we note from App. C.5.2 that the boost operator of the $(1,0)$ representation is given by

$$B^{(1,0)} \equiv \exp[i\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}}] = \mathbb{1}_3 + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\left[\sinh(\varphi)\right] + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\left[\cosh(\varphi)-1\right], \tag{4.1.8}$$

with $\boldsymbol{K} \equiv -i\boldsymbol{J}$. Similarly, the boost operator of the $(0,1)$ representation is given by

$$B^{(0,1)} \equiv \exp[i\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}}] = \mathbb{1}_3 + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\left[\sinh(\varphi)\right] + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\left[\cosh(\varphi)-1\right], \tag{4.1.9}$$

with $\boldsymbol{K} \equiv +i\boldsymbol{J}$. In both (4.1.8) and (4.1.9), φ is the rapidity parameter defined in the usual fashion by

$$\sinh(\varphi) = |\boldsymbol{p}|/m \quad \text{and} \quad \cosh(\varphi) = p^0/m.$$

The unit rapidity vector $\hat{\boldsymbol{\varphi}}$ is chosen to be equal to the unit three-momentum $\hat{\boldsymbol{p}}$.

The boost operator $D[L(p)] \equiv B^{(1,1)}$ of the $(1,0)\otimes(0,1)$ representation is given in terms of (4.1.8) and (4.1.9) by

$$D[L(p)] \equiv B^{(1,0)} \otimes B^{(0,1)}. \tag{4.1.10}$$

Computing the tensor product and subsequently using

$$\mathscr{K}_k = \frac{1}{i}\frac{\partial D[L(p)]}{\partial \varphi_k}\bigg|_{\varphi_k \to 0}, \tag{4.1.11}$$

we obtain the boost generators $\mathscr{K}_x$, $\mathscr{K}_y$, and $\mathscr{K}_z$, respectively, as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1.12}$$

Having thus derived the generators of rotation and boost of the $(1,1)$ representation, we now apply the constraints of Sec. 2.5.3 to determine the explicit form of the coefficient functions $\boldsymbol{u}(p;\sigma,s)$ and $\boldsymbol{v}(p;\sigma,s)$.

4.2 Coefficient functions

The coefficient functions at rest are obtained in the first instance from the rotation constraints derived in Sec. 2.5.3. The relevant expression (2.5.58) reads

$$U\boldsymbol{J} = \boldsymbol{\mathscr{J}}U \quad \text{and} \quad -V\boldsymbol{J}^* = \boldsymbol{\mathscr{J}}V. \tag{4.2.1}$$

Here $\boldsymbol{\mathscr{J}} \equiv (\mathscr{J}_x, \mathscr{J}_y, \mathscr{J}_z)$, the elements being those derived above[1] and given explicitly in (4.1.7); the matrices U and V, defined in (2.5.53)–(2.5.57), are composed of the coefficient functions at rest $\boldsymbol{u}(k;\sigma,s)$ and $\boldsymbol{v}(k;\sigma,s)$, respectively; finally, the matrix $\boldsymbol{J} \equiv \boldsymbol{J}^{(0)} \oplus \boldsymbol{J}^{(1)} \oplus \boldsymbol{J}^{(2)}$ is obtained via (2.3.30) and has components

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \sqrt{\frac{3}{2}} & 0 & \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.2.2}$$

[1] It is here that the advantage in the above choice of the adjoint representation becomes apparent. Unlike the rotation generators of the standard representation, the generators of the adjoint representation have purely imaginary components. As a consequence, the non-zero components of the matrices $\mathscr{J}_k$ have turned out to be purely imaginary. This is favourable in that the solution of (4.2.1) for the matrix V is simply the complex conjugate of the matrix U.

$$J_y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{3}{2}} & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \end{pmatrix}, \tag{4.2.3}$$

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{4.2.4}$$

Using the above two representations of the Lie algebra of the rotation group, we solve (4.2.1) and find the coefficient functions as follows. For $\boldsymbol{u}(k;\sigma,s)$, we have nine coefficient functions. Those with spin labels zero and one, $\boldsymbol{u}(k;0,0)$, $\boldsymbol{u}(k;1,1)$, $\boldsymbol{u}(k;0,1)$, and $\boldsymbol{u}(k;-1,1)$ read, respectively,

$$\begin{pmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ c_1 \\ 0 \\ 0 \\ 0 \\ c_1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -c_2 \\ 0 \\ 0 \\ -ic_2 \\ c_2 \\ ic_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ i\sqrt{2}\,c_2 \\ 0 \\ -i\sqrt{2}\,c_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -c_2 \\ 0 \\ 0 \\ ic_2 \\ c_2 \\ -ic_2 \\ 0 \end{pmatrix}. \tag{4.2.5}$$

Furthermore, for spin label two, we have $\boldsymbol{u}(k;2,2)$, $\boldsymbol{u}(k;1,2)$, $\boldsymbol{u}(k;0,2)$, $\boldsymbol{u}(k;-1,2)$, and $\boldsymbol{u}(k;-2,2)$:

$$\begin{pmatrix} c_3 \\ ic_3 \\ 0 \\ ic_3 \\ -c_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -c_3 \\ 0 \\ 0 \\ -ic_3 \\ -c_3 \\ -ic_3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\sqrt{\frac{2}{3}}\,c_3 \\ 0 \\ 0 \\ 0 \\ -\sqrt{\frac{2}{3}}\,c_3 \\ 0 \\ 0 \\ 0 \\ 2\sqrt{\frac{2}{3}}\,c_3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ c_3 \\ 0 \\ 0 \\ -ic_3 \\ c_3 \\ -ic_3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} c_3 \\ -ic_3 \\ 0 \\ -ic_3 \\ -c_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2.6}$$

Similarly, for $\boldsymbol{v}(k;\sigma,s)$, we have $\boldsymbol{v}(k;0,0)$, $\boldsymbol{v}(k;1,1)$, $\boldsymbol{v}(k;0,1)$, and $\boldsymbol{v}(k;-1,1)$ given, respectively, by

$$\begin{pmatrix} c_4 \\ 0 \\ 0 \\ 0 \\ c_4 \\ 0 \\ 0 \\ 0 \\ c_4 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -c_5 \\ 0 \\ 0 \\ ic_5 \\ c_5 \\ -ic_5 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -i\sqrt{2}\,c_5 \\ 0 \\ i\sqrt{2}\,c_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -c_5 \\ 0 \\ 0 \\ -ic_5 \\ c_5 \\ ic_5 \\ 0 \end{pmatrix}. \tag{4.2.7}$$

Lastly, $\boldsymbol{v}(k;2,2)$, $\boldsymbol{v}(k;1,2)$, $\boldsymbol{v}(k;0,2)$, $\boldsymbol{v}(k;-1,2)$, and $\boldsymbol{v}(k;-2,2)$ are given, respectively, by

$$\begin{pmatrix} c_6 \\ -ic_6 \\ 0 \\ -ic_6 \\ -c_6 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -c_6 \\ 0 \\ 0 \\ ic_6 \\ -c_6 \\ ic_6 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\sqrt{\frac{2}{3}}\,c_6 \\ 0 \\ 0 \\ 0 \\ -\sqrt{\frac{2}{3}}\,c_6 \\ 0 \\ 0 \\ 0 \\ 2\sqrt{\frac{2}{3}}\,c_6 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ c_6 \\ 0 \\ 0 \\ ic_6 \\ c_6 \\ ic_6 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} c_6 \\ ic_6 \\ 0 \\ ic_6 \\ -c_6 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2.8}$$

The eighteen coefficient functions $\boldsymbol{u}(k;\sigma,s)$ and $\boldsymbol{v}(k;\sigma,s)$ at rest are thus given in terms of six complex parameters c_i, $i \in \{1,2,\ldots,6\}$. These will be further constrained in the sections to follow. The validity of the labelling of the coefficient functions at rest in terms of the eigenvalues of $\mathscr{J}_z$ and $\mathscr{J}^2 \equiv \mathscr{J}_x^2 + \mathscr{J}_y^2 + \mathscr{J}_z^2$ is confirmed by

$$\mathscr{J}_z\, \boldsymbol{u}(k;\sigma,s) = \sigma\, \boldsymbol{u}(k;\sigma,s), \qquad \mathscr{J}^2 \boldsymbol{u}(k;\sigma,s) = s\,(s+1)\, \boldsymbol{u}(k;\sigma,s), \tag{4.2.9}$$

and

$$\mathscr{J}_z\, \boldsymbol{v}(k;\sigma,s) = -\sigma\, \boldsymbol{v}(k;\sigma,s), \qquad \mathscr{J}^2 \boldsymbol{v}(k;\sigma,s) = s\,(s+1)\, \boldsymbol{v}(k;\sigma,s). \tag{4.2.10}$$

With the coefficient functions thus defined at rest they may be computed at arbitrary momentum p^μ by

$$u_l(p;\sigma,s) = \sqrt{\frac{m}{p^0}} \sum_{\bar{l}} D_{l\bar{l}}[L(p)] u_{\bar{l}}(k;\sigma,s), \tag{4.2.11}$$

$$v_l(p;\sigma,s) = \sqrt{\frac{m}{p^0}} \sum_{\bar{l}} D_{l\bar{l}}[L(p)] v_{\bar{l}}(k;\sigma,s), \tag{4.2.12}$$

where $D[L(p)]$ is as defined above in (4.1.10).

4.3 The dual quantum field

Ere we attempt to further constrain the free parameters in the components of $\psi(x)$, it will prove convenient to first derive the dual field, the metric, and the dual coefficient functions. As in the example of Ch. 3, the dual quantum field $\bar{\psi}(x)$ is related to the field $\psi(x)$ by Hermitian conjugation and a conjugate-linear mapping of the coefficient functions. It reads

$$\bar{\psi}_l(x) = \sum_{\sigma,s} \int d^3p\, (2\pi)^{-3/2} \Big[\kappa^*\, \mathrm{e}^{+ip\cdot x}\, \bar{u}_l(p;\sigma,s) a^\dagger(p;\sigma,s) \\ + \lambda^*\, \mathrm{e}^{-ip\cdot x}\, \bar{v}_l(p;\sigma,s) b(p;\sigma,s)\Big], \tag{4.3.1}$$

where $a^\dagger(p;\sigma,s)$ and $b(p;\sigma,s)$ are particle creation and antiparticle annihilation operators, respectively; $\bar{\boldsymbol{u}}(p;\sigma,s)$ and $\bar{\boldsymbol{v}}(p;\sigma,s)$ are the dual coefficient functions defined as

$$\bar{u}_l(p;\sigma,s) \equiv \sum_{\bar{l}} \eta_{l\bar{l}}(s) u^\dagger_{\bar{l}}(p;\sigma,s), \tag{4.3.2}$$

$$\bar{v}_l(p;\sigma,s) \equiv \sum_{\bar{l}} \eta_{l\bar{l}}(s) v^\dagger_{\bar{l}}(p;\sigma,s), \tag{4.3.3}$$

where $\eta(s)$ is a square non-singular matrix, the metric, which we now derive. Recalling the discussion of Sec. 3.3 and the general treatment of App. B.5, we begin by imposing invariance of the sesquilinear form under $D[\Lambda]$, as expressed in the constraints (B.5.16).

These read

$$\{\mathscr{K}_i, \eta\} = 0 \quad \text{and} \quad [\mathscr{J}_i, \eta] = 0, \quad \text{for} \quad i \in \{x, y, z\}\,. \tag{4.3.4}$$

Taking a complex valued 9×9 matrix and imposing (4.3.4), we obtain

$$\eta = \begin{pmatrix} \varpi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varpi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varpi & 0 & 0 \\ 0 & \varpi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varpi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varpi & 0 \\ 0 & 0 & \varpi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varpi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varpi \end{pmatrix}, \tag{4.3.5}$$

where $\varpi \in \mathbb{C}$. As was discussed at some length in Sec. 3.3, it is mathematically consistent with the definition of a sesquilinear form to allow ϖ to depend upon the indices s and σ. Nevertheless, as was shown also in Sec. 3.3, a dependence on σ is excluded by (3.3.8), the counterpart of the rotation constraints (4.2.1). We are thus left with a functional dependence upon spin index s. Accordingly, the metric takes the form

$$\eta(s) = \varpi(s)\beta, \tag{4.3.6}$$

where β is the constant matrix implicitly defined by (4.3.5) together with (4.3.6). We will demand, for the sake of simplicity, that the metric be involutory; that is, $\eta(s)\eta(s) = \mathbb{1}$. With this requirement, the functions $\varpi(s)$ are restricted to take on values of ± 1. The remaining freedom in the functional dependence of the metric will be constrained in the next section.

4.4 The propagator and the consistency and unitarity of quantum field theory

In order that the theory here under consideration be consistent and unitary at all energies without the need for regulator terms, it is sufficient that the propagator be the Feynman propagator for spin two. From the discussion of Sec. 2.6, we know that this requirement places the following constraints upon the coefficient functions at rest:

$$\sum_{\sigma,s} \boldsymbol{u}(k;\sigma,s)\boldsymbol{u}^\dagger(k;\sigma,s)\varpi(s) = \frac{1}{|\kappa|^2 2m}\beta, \tag{4.4.1}$$

$$\sum_{\sigma,s} \boldsymbol{v}(k;\sigma,s)\boldsymbol{v}^\dagger(k;\sigma,s)\varpi(s) = \frac{1}{|\lambda|^2 2m}\beta. \tag{4.4.2}$$

Given that the overall scale of the coefficient functions is not yet fixed, we may choose, without loss of generality, $|\kappa| = |\lambda| = 1$. For future use, we write these explicitly in terms of their magnitude and a phase:

$$\kappa = \mathrm{e}^{i\phi_7} \quad \text{and} \quad \lambda = \mathrm{e}^{i\phi_8}, \tag{4.4.3}$$

where $\phi_7, \phi_8 \in \mathbb{R}$. Thus, computing the spin sums and imposing (4.4.1), we find

$$3|c_1|^2\varpi(0) = -4|c_2|^2\varpi(1) = 4|c_3|^2\varpi(2) = \frac{1}{2m}. \tag{4.4.4}$$

Repeating this exercise for (4.4.2), we obtain

$$3|c_4|^2\varpi(0) = -4|c_5|^2\varpi(1) = 4|c_6|^2\varpi(2) = \frac{1}{2m}. \tag{4.4.5}$$

As mentioned at the end of the previous section, we have chosen the metric to be involutory, thus placing the restriction $\varpi(s) = \pm 1$. This, together with (4.4.4) or (4.4.5), yields

$$\varpi(0) = 1, \quad \varpi(1) = -1, \quad \text{and} \quad \varpi(2) = 1. \tag{4.4.6}$$

With this, the metric $\eta(s)$ is completely constrained. Furthermore, substituting (4.4.6) back into (4.4.4) and (4.4.5), and rewriting the parameters c_i in terms of their magnitude and a phase $\mathrm{e}^{i\phi_i}$, where $\phi_i \in \mathbb{R}$ with $i \in \{1, 2, \dots, 6\}$, we obtain

$$c_1 = \frac{1}{\sqrt{6m}}\,\mathrm{e}^{i\phi_1}, \quad c_2 = \frac{1}{\sqrt{8m}}\,\mathrm{e}^{i\phi_2}, \quad c_3 = \frac{1}{\sqrt{8m}}\,\mathrm{e}^{i\phi_3}, \tag{4.4.7}$$

$$c_4 = \frac{1}{\sqrt{6m}}\,\mathrm{e}^{i\phi_4}, \quad c_5 = \frac{1}{\sqrt{8m}}\,\mathrm{e}^{i\phi_5}, \quad c_6 = \frac{1}{\sqrt{8m}}\,\mathrm{e}^{i\phi_6}\,. \tag{4.4.8}$$

The constraints (4.4.1) and (4.4.2) are hereby satisfied. Invoking the identity

$$D[L(p)]\beta D[L(p)]^\dagger = \beta,$$

we obtain the following spin sums at momentum $\boldsymbol{p}$:

$$N(\boldsymbol{p}) \equiv \sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s)\boldsymbol{u}^\dagger(p;\sigma,s)\beta\varpi(s) = \frac{1}{2p^0}\mathbb{1}_9, \tag{4.4.9}$$

$$M(\boldsymbol{p}) \equiv \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s)\boldsymbol{v}^\dagger(p;\sigma,s)\beta\varpi(s) = \frac{1}{2p^0}\mathbb{1}_9, \tag{4.4.10}$$

where we have multiplied on both sides by β and used the involutory nature of this matrix to obtain the 9×9 identity matrix $\mathbb{1}_9$ on the RHS. Substituting these into the general expression (B.4.11), we obtain, as desired, the propagator

$$\Delta_{F2}(x - x') = (2\pi)^{-4} \int d^4\rho\, \mathrm{e}^{-i\rho\cdot(x-x')} \left[\frac{\mathbb{1}_9}{-\rho^2 + m^2 - i\epsilon}\right], \tag{4.4.11}$$

that is, the Feynman propagator for spin two.

4.5 Discrete symmetries

Having thus far constrained the metric and fixed the magnitude of the remaining free parameters, we now impose the constraints that arise in the context of the discrete symmetries. As in the previous chapter, we denote the respective matrix operators for space-inversion and time-reversal on the coefficient functions as $D[\mathscr{P}]$ and $D[\mathscr{T}]$, rather than $D[\mathscr{P}^{-1}]$ and $D[\mathscr{T}^{-1}]$, for notational simplicity.

4.5.1 Space-inversion

Availing ourselves of the constraints on the coefficient functions at rest, as derived in Sec. 2.5.4, we have the following from (2.5.65) and (2.5.66):

$$D[\mathscr{P}]\boldsymbol{u}(k;\sigma,s) = \xi_s^* \boldsymbol{u}(k;\sigma,s), \tag{4.5.1}$$

$$D[\mathscr{P}]\boldsymbol{v}(k;\sigma,s) = \xi_s^c \boldsymbol{v}(k;\sigma,s), \tag{4.5.2}$$

where, by (2.5.31), the space-inversion matrix $D[\mathscr{P}]$ is subject to

$$D[\mathscr{P}]\mathscr{J}_i D^{-1}[\mathscr{P}] = +\mathscr{J}_i \quad \text{and} \quad D[\mathscr{P}]\mathscr{K}_i D^{-1}[\mathscr{P}] = -\mathscr{K}_i, \tag{4.5.3}$$

for $i \in \{x, y, z\}$. These constraints are manifestly equivalent to those employed above in (4.3.4) to derive the matrix η. Therefore, we have

$$D[\mathscr{P}] = \varrho\beta, \tag{4.5.4}$$

where $\varrho \in \mathbb{C}$. Applying $D[\mathscr{P}]$ to the coefficient functions, we obtain

$$D[\mathscr{P}]\boldsymbol{u}(k;\sigma,s) = (-)^s \varrho\, \boldsymbol{u}(k;\sigma,s), \tag{4.5.5}$$

$$D[\mathscr{P}]\boldsymbol{v}(k;\sigma,s) = (-)^s \varrho\, \boldsymbol{v}(k;\sigma,s). \tag{4.5.6}$$

Hence, the constraints (4.5.5) and (4.5.6) will be met provided

$$\xi_s^* = \xi_s^c = (-)^s \varrho. \tag{4.5.7}$$

This shows that ϱ must be of absolute value one. Furthermore, particles and antiparticles in this theory are of the same intrinsic parity, however, there is a difference in sign between the spin one sector and the sectors of spin zero and spin two.

4.5.2 Time-reversal

The constraints on the coefficient functions at rest under the action of space-inversion were derived in Sec. 2.5.4. From (2.5.71) and (2.5.72), with κ and λ given by (4.4.3), the constraints read

$$D[\mathscr{T}]\boldsymbol{u}(k;\sigma,s) = \mathrm{e}^{-2i\phi_7}\,\zeta_s^*(-)^{s+\sigma}\boldsymbol{u}^*(k;-\sigma,s), \tag{4.5.8}$$

$$D[\mathscr{T}]\boldsymbol{v}(k;\sigma,s) = \mathrm{e}^{-2i\phi_8}\,\zeta_s^c(-)^{s+\sigma}\boldsymbol{v}^*(k;-\sigma,s). \tag{4.5.9}$$

Looking at (2.5.32), we find that the time-reversal matrix $D[\mathscr{T}]$ must satisfy

$$D[\mathscr{T}]\mathscr{J}_i D^{-1}[\mathscr{T}] = -\mathscr{J}_i^* \quad \text{and} \quad D[\mathscr{T}]\mathscr{K}_i D^{-1}[\mathscr{T}] = +\mathscr{K}_i^*, \tag{4.5.10}$$

for $i \in \{x,y,z\}$. Solving for $D[\mathscr{T}]$, using the generators as given above in (4.1.7) and (4.1.12), we find

$$D[\mathscr{T}] = \varsigma\mathbb{1}_9, \tag{4.5.11}$$

where $\varsigma \in \mathbb{C}$. Applying this to $\boldsymbol{u}(k;\sigma,s)$, we obtain

$$D[\mathscr{T}]\boldsymbol{u}(k;0,0) = \varsigma\,\mathrm{e}^{2i\phi_1}\,\boldsymbol{u}^*(k;0,0), \tag{4.5.12}$$

$$D[\mathscr{T}]\boldsymbol{u}(k;\sigma,1) = \varsigma\,\mathrm{e}^{2i\phi_2}(-)^{1+\sigma}\boldsymbol{u}^*(k;-\sigma,1), \tag{4.5.13}$$

$$D[\mathscr{T}]\boldsymbol{u}(k;\sigma,2) = \varsigma\,\mathrm{e}^{2i\phi_3}(-)^{2+\sigma}\boldsymbol{u}^*(k;-\sigma,2). \tag{4.5.14}$$

Similarly, for $\boldsymbol{v}(k;\sigma,s)$, we have

$$D[\mathscr{T}]\boldsymbol{v}(k;0,0) = \varsigma\,\mathrm{e}^{2i\phi_4}\,\boldsymbol{v}^*(k;0,0), \tag{4.5.15}$$

$$D[\mathscr{T}]\boldsymbol{v}(k;\sigma,1) = \varsigma\,\mathrm{e}^{2i\phi_5}(-)^{1+\sigma}\boldsymbol{v}^*(k;-\sigma,1), \tag{4.5.16}$$

$$D[\mathscr{T}]\boldsymbol{v}(k;\sigma,2) = \varsigma\,\mathrm{e}^{2i\phi_6}(-)^{2+\sigma}\boldsymbol{v}^*(k;-\sigma,2). \tag{4.5.17}$$

Hence, in order that the constraints (4.5.8) and (4.5.9) be satisfy, the following phase relations must respected:

$$\zeta_0^* = \varsigma\,\mathrm{e}^{2i\phi_1+2i\phi_7}, \quad \zeta_1^* = \varsigma\,\mathrm{e}^{2i\phi_2+2i\phi_7}, \quad \zeta_2^* = \varsigma\,\mathrm{e}^{2i\phi_3+2i\phi_7}, \tag{4.5.18}$$

$$\zeta_0^c = \varsigma\,\mathrm{e}^{2i\phi_4+2i\phi_8}, \quad \zeta_1^c = \varsigma\,\mathrm{e}^{2i\phi_5+2i\phi_8}, \quad \zeta_2^c = \varsigma\,\mathrm{e}^{2i\phi_6+2i\phi_8}\,. \tag{4.5.19}$$

This shows that ς must be of modulus one. No restrictions are placed upon the relative signs of the time-reversal phases.

4.5.3 Charge-conjugation

The constraints on the coefficient functions at rest were derived in Sec. 2.5.4. From (2.5.92) and (2.5.93), using κ and λ as given in (4.4.3), the constraints read

$$A\boldsymbol{u}^*(k;\sigma,s) = \mathrm{e}^{i\phi_7+i\phi_8}\,\xi_s^{c*}\zeta_s^{c*}\boldsymbol{v}(k;\sigma,s), \tag{4.5.20}$$

$$A\boldsymbol{v}^*(k;\sigma,s) = \mathrm{e}^{i\phi_7+i\phi_8}\,\xi_s\zeta_s\,\boldsymbol{u}(k;\sigma,s). \tag{4.5.21}$$

Inserting ξ and ζ from (4.5.7), (4.5.18), and (4.5.19), we have, for $s = 0$,

$$A\boldsymbol{u}^*(k;0,0) = \mathrm{e}^{i\phi_7 - i\phi_8 - 2i\phi_4}\, \varrho^* \varsigma^* \boldsymbol{v}(k;0,0), \tag{4.5.22}$$

$$A\boldsymbol{v}^*(k;0,0) = \mathrm{e}^{-i\phi_7 + i\phi_8 - 2i\phi_1}\, \varrho^* \varsigma^* \boldsymbol{u}(k;0,0). \tag{4.5.23}$$

Similarly, for $s = 1$, we have

$$A\boldsymbol{u}^*(k;0,1) = -\,\mathrm{e}^{i\phi_7 - i\phi_8 - 2i\phi_5}\, \varrho^* \varsigma^* \boldsymbol{v}(k;0,1), \tag{4.5.24}$$

$$A\boldsymbol{v}^*(k;0,1) = -\,\mathrm{e}^{-i\phi_7 + i\phi_8 - 2i\phi_2}\, \varrho^* \varsigma^* \boldsymbol{u}(k;0,1). \tag{4.5.25}$$

Finally, for $s = 2$, we have

$$A\boldsymbol{u}^*(k;0,2) = \mathrm{e}^{i\phi_7 - i\phi_8 - 2i\phi_6}\, \varrho^* \varsigma^* \boldsymbol{v}(k;0,2), \tag{4.5.26}$$

$$A\boldsymbol{v}^*(k;0,2) = \mathrm{e}^{-i\phi_7 + i\phi_8 - 2i\phi_3}\, \varrho^* \varsigma^* \boldsymbol{u}(k;0,2). \tag{4.5.27}$$

In accordance with the result given in (2.5.90), the charge-conjugation matrix A is derived via

$$A\,\mathscr{J}_i^*A^{-1} = -\,\mathscr{J}_i \quad \text{and} \quad A\mathscr{K}_i^*A^{-1} = -\mathscr{K}_i, \tag{4.5.28}$$

for $i \in \{x,y,z\}$. Looking at the explicit form of the matrices $\mathscr{J}_i$ and $\mathscr{K}_i$, as found in (4.1.7) and (4.1.12) above, we note that these are completely imaginary and completely real, respected. Hence, the constraints (4.5.28) are identical to the constraints given above in (4.3.4) for the matrix η. We thus conclude

$$A = \delta\beta, \tag{4.5.29}$$

where $\delta \in \mathbb{C}$. Applying this to $\boldsymbol{u}(k;\sigma,s)$, we obtain

$$A\boldsymbol{u}^*(k;0,0) = +\delta\,\mathrm{e}^{-i\phi_1 - i\phi_4}\, \boldsymbol{v}(k;0,0), \tag{4.5.30}$$

$$A\boldsymbol{u}^*(k;\sigma,1) = -\delta\,\mathrm{e}^{-i\phi_2 - i\phi_5}\, \boldsymbol{v}(k;\sigma,1), \tag{4.5.31}$$

$$A\boldsymbol{u}^*(k;\sigma,2) = +\delta\,\mathrm{e}^{-i\phi_3 - i\phi_6}\, \boldsymbol{v}(k;\sigma,2). \tag{4.5.32}$$

Likewise, for $\boldsymbol{v}(k;\sigma,s)$, we find

$$A\boldsymbol{v}^*(k;0,0) = +\delta\,\mathrm{e}^{-i\phi_1 - i\phi_4}\, \boldsymbol{u}(k;0,0), \tag{4.5.33}$$

$$A\boldsymbol{v}^*(k;\sigma,1) = -\delta\,\mathrm{e}^{-i\phi_2 - i\phi_5}\, \boldsymbol{u}(k;\sigma,1), \tag{4.5.34}$$

$$A\boldsymbol{v}^*(k;\sigma,2) = +\delta\,\mathrm{e}^{-i\phi_3 - i\phi_6}\, \boldsymbol{u}(k;\sigma,2). \tag{4.5.35}$$

Hence, in order that the constraints (4.5.22)–(4.5.27) be satisfied, it is necessary and sufficient to demand that the following phase relationships be respected:

$$\varrho\varsigma\delta = \mathrm{e}^{+i\phi_1 - i\phi_4 + i\phi_7 - i\phi_8}, \tag{4.5.36}$$

$$\varrho\varsigma\delta = \mathrm{e}^{-i\phi_1 + i\phi_4 - i\phi_7 + i\phi_8}, \tag{4.5.37}$$

$$\varrho\varsigma\delta = \mathrm{e}^{+i\phi_2 - i\phi_5 + i\phi_7 - i\phi_8}, \tag{4.5.38}$$

$$\varrho\varsigma\delta = \mathrm{e}^{-i\phi_2 + i\phi_5 - i\phi_7 + i\phi_8}, \tag{4.5.39}$$

$$\varrho\varsigma\delta = \mathrm{e}^{+i\phi_3 - i\phi_6 + i\phi_7 - i\phi_8}, \tag{4.5.40}$$

$$\varrho\varsigma\delta = \mathrm{e}^{-i\phi_3 + i\phi_6 - i\phi_7 + i\phi_8}. \tag{4.5.41}$$

Recalling that both ϱ and ς have been found to be of modulus one, it follows directly from any one of the here given phase relationships that δ must also be of modulus one. Furthermore, taking (4.5.36) together with (4.5.37), it is clear that $\varrho\varsigma\delta = \pm 1$.

4.5.4 CPT

Before we move on to the next section, we briefly explore the implication of the above results upon the transformation property of the field under the succession of discrete symmetries CPT. From (2.5.98), we have

$$(CPT)\,\psi(x)\,(CPT)^{-1} = D[\mathscr{T}]D[\mathscr{P}]A\,\psi^*(\mathscr{P}\mathscr{T}x). \tag{4.5.42}$$

It is immediately clear by inspection of the explicit form of matrices on the RHS of (3.5.34), as given above in (4.5.4), (4.5.11), and (4.5.29), that $D[\mathscr{T}]D[\mathscr{P}]A = \varrho\varsigma\delta\mathbb{1}$. We thus obtain the following transformation property of the field under CPT:

$$(CPT)\,\psi(x)\,(CPT)^{-1} = \varrho\varsigma\delta\,\psi^*(-x), \tag{4.5.43}$$

where, by the constraints summarised in (4.5.36)–(4.5.41), the phase $\varrho\varsigma\delta = \pm 1$, depending upon our choice of parameters $\phi_1, \phi_2, \ldots, \phi_8$. This result confirms the consistency of the above construct with the CPT theorem [40, Sec. 5.8].

4.6 Field commutators

As discussed in Ch. 2, it follows in the Weinberg formalism from the demand that the S-matrix be Poincaré invariant that the field and its dual must satisfy the following commutation relations:

$$[\psi_l(x), \psi_{\bar{l}}(y)]_- = 0, \tag{4.6.1}$$

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_- = 0, \tag{4.6.2}$$

for $(x-y)^2 < 0$. We now evaluate both of these in turn, for the case of a distinct as for that of an indistinct antiparticle, and thereby explore what further constraints may result on the parameters $\phi_1, \phi_2, \ldots, \phi_8$.

We begin with the evaluation of the second of the above commutators because this is identical in the case of a distinct antiparticle as compared to that of an indistinct antiparticle. Expanding the LHS of (4.6.2) using the bosonic commutation relations of the creation and annihilation operators, as given in (2.4.3) and (2.4.4), we find

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_- = \int \frac{d^3p}{(2\pi)^3} \left[N_{l\bar{l}}(\boldsymbol{p})\, \mathrm{e}^{-ip\cdot(x-y)} - M_{l\bar{l}}(\boldsymbol{p})\, \mathrm{e}^{+ip\cdot(x-y)} \right], \tag{4.6.3}$$

where $N(\boldsymbol{p})$ and $M(\boldsymbol{p})$ are the spin sums which, in (4.4.9) and (4.4.10), were evaluated as $2p^0 N_{l\bar{l}}(\boldsymbol{p}) = 2p^0 M_{l\bar{l}}(\boldsymbol{p}) = \delta^l{}_{\bar{l}}$, a 9×9 identity matrix in component form. Substitution into (4.6.3) yields

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_- = \delta^l{}_{\bar{l}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left[\mathrm{e}^{-ip\cdot(x-y)} - \mathrm{e}^{+ip\cdot(x-y)} \right]. \tag{4.6.4}$$

This can be expressed in terms of the standard function $\Delta_+(x)$ defined in (3.6.6). We thus obtain

$$[\psi_l(x), \bar{\psi}_{\bar{l}}(y)]_- = \delta^l{}_{\bar{l}}\, \Delta(x-y), \tag{4.6.5}$$

where $(\delta^l{}_{\bar{l}}) = \mathbb{1}_9$; $\Delta(x-y) \equiv \Delta_+(x-y) - \Delta_+(y-x)$, as given in (3.6.5). From (3.6.7), we recall that $\Delta(x-y)$ vanishes for a space-like argument; therefore, (4.6.5) vanishes for $(x-y)^2 < 0$, as desired.

Turning now to the remaining commutator, we know from Sec. 2.5.5 that (4.6.1) vanishes in the case of a distinct antiparticle on account of the commutation relations of the creation and annihilation operators. In the alternate scenario, we have $a(p;\sigma,s) \equiv b(p;\sigma,s)$ and the LHS of (4.6.1) is expanded, as in (2.5.102), to read

$$\begin{aligned} [\psi_l(x), \psi_{\bar{l}}(y)]_- = \kappa\lambda \sum_{\sigma,s} \int \frac{d^3p}{(2\pi)^3} \Big[& u_l(p;\sigma,s) v_{\bar{l}}(p;\sigma,s)\, \mathrm{e}^{-ip\cdot(x-y)} \\ & - v_l(p;\sigma,s) u_{\bar{l}}(p;\sigma,s)\, \mathrm{e}^{+ip\cdot(x-y)} \Big]. \end{aligned} \tag{4.6.6}$$

With the phase choice

$$\mathrm{e}^{i(\phi_1+\phi_4)} = \mathrm{e}^{i(\phi_2+\phi_5)} = \mathrm{e}^{i(\phi_3+\phi_6)} = 1, \tag{4.6.7}$$

the spin sums are computed to be

$$\sum_{\sigma,s} u_l(p;\sigma,s) v_{\bar{l}}(p;\sigma,s) = \sum_{\sigma,s} v_l(p;\sigma,s) u_{\bar{l}}(p;\sigma,s) = \frac{1}{2p^0} \delta^l{}_{\bar{l}}. \tag{4.6.8}$$

Substituting (4.6.8) into (4.6.6), we obtain

$$[\psi_l(x), \psi_{\bar{l}}(y)]_- = \kappa\lambda\, \delta^l{}_{\bar{l}}\, \Delta(x-y). \tag{4.6.9}$$

Therefore, by (3.6.7),

$$[\psi_l(x), \psi_{\bar{l}}(y)]_- = 0, \quad \text{for} \quad (x-y)^2 < 0, \tag{4.6.10}$$

as required.

4.7 Field commutators and discrete symmetries

In the particular case in which the metric is dependent upon the spin index s, the constraints (4.5.20) and (4.5.20) imposed above on the coefficient functions do not immediately guarantee that the charge-conjugation transformed field will commute with $\psi(x)$ and with $\bar{\psi}(x)$ at space-like separations. We must check two further conditions, as given in (2.5.87) and (2.5.88). For a bosonic theory with distinct or indistinct antiparticles, we must have

$$\sum_{\sigma,s} \int \frac{d^3p}{(2\pi)^3} \Big[\boldsymbol{u}(p;\sigma,s)\boldsymbol{u}^\dagger(p;\sigma,s)\, \mathrm{e}^{-ip\cdot(x-y)} \\ -\boldsymbol{v}(p;\sigma,s)\boldsymbol{v}^\dagger(p;\sigma,s)\, \mathrm{e}^{+ip\cdot(x-y)}\Big] = 0, \tag{4.7.1}$$

for $(x-y)^2 < 0$. For a bosonic theory with an indistinct antiparticle, we also require

$$\sum_{\sigma,s} \int \frac{d^3p}{(2\pi)^3} \Big[\boldsymbol{u}(p;\sigma,s)\boldsymbol{v}^T(p;\sigma,s)\eta(s)\, \mathrm{e}^{-ip\cdot(x-y)} \\ -\boldsymbol{v}(p;\sigma,s)\boldsymbol{u}^T(p;\sigma,s)\eta(s)\, \mathrm{e}^{+ip\cdot(x-y)}\Big] = 0, \tag{4.7.2}$$

for $(x-y)^2 < 0$. The terms $|\kappa|^2$ and $|\lambda|^2$ that appear in (2.5.87) have here been set to one, as per (4.4.3).

Evaluating the spin sums in (4.7.1), we find that they admit the following manifestly covariant form:

$$\sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s)\boldsymbol{u}^\dagger(p;\sigma,s) = \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s)\boldsymbol{v}^\dagger(p;\sigma,s) \\ = \frac{1}{2p^0 m^4} S^{-1}\left(\gamma_{\mu\nu\alpha\beta}p^\mu p^\nu p^\alpha p^\beta + m^4 \Upsilon\right) S\beta, \tag{4.7.3}$$

where $\gamma_{\mu\nu\alpha\beta}$ is a symmetric traceless tensor of rank 4; Υ is equal to a direct sum given by

$\mathbb{1}_1 \oplus -\mathbb{1}_3 \oplus \mathbb{1}_5$, where $\mathbb{1}_n$ is an $n \times n$ identity matrix; S is the given by

$$S = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.7.4}$$

Substituting (4.7.3) into the LHS of (4.7.1), the resulting expression can be written as

$$S^{-1}\left(-\frac{\gamma_{\mu\nu\alpha\beta}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{m^4} + \Upsilon\right) S\beta\Delta(x-y).$$

This vanishes by (3.6.7), as desired, for all space-like separated x and y. As promised in Sec. 2.5.4, this result shows that (4.7.1) is satisfied without imposing any further constraints on the parameters $\phi_1, \phi_2, \ldots, \phi_8$.

Turning now to the second of the above constraints, we again begin by evaluating the spin sums. Given that this constraint arises only in the case of an indistinct antiparticle and recalling the choice of phases (4.6.7), we find

$$\begin{aligned} \sum_{\sigma,s} \boldsymbol{u}(p;\sigma,s)\bar{\boldsymbol{v}}^*(p;\sigma,s) &= \sum_{\sigma,s} \boldsymbol{v}(p;\sigma,s)\bar{\boldsymbol{u}}^*(p;\sigma,s) \\ &= \frac{1}{2p^0 m^4} S^{-1}\left(\gamma_{\mu\nu\alpha\beta}p^\mu p^\nu p^\alpha p^\beta + m^4\Upsilon\right) S\beta. \end{aligned} \tag{4.7.5}$$

Substitution into the LHS of (4.7.2) then yields

$$S^{-1}\left(-\frac{\gamma_{\mu\nu\alpha\beta}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{m^4} + \Upsilon\right) S\beta\Delta(x-y).$$

Again, by (3.6.7), this vanishes for $(x-y)^2 < 0$. The condition (4.7.2) is thus met without any further constraints on the parameters $\phi_1, \phi_2, \ldots, \phi_8$.

The spin sums computed above in (4.7.3) turn out to be identical to (4.7.5). It is a consequence of the identity

$$\boldsymbol{u}(p;\sigma,s) = \eta(s)\boldsymbol{v}^*(p;\sigma,s), \tag{4.7.6}$$

which holds provided the phases are chosen as in (4.6.7). This can be easily verified by checking (4.7.6) at rest and subsequently using the identity $\beta D[L(p)]\beta = D^*[L(p)]$, obtained from (2.5.89) along with (4.5.29), to return to an expression in terms of coefficient functions at arbitrary momentum.

4.8 The canonical formalism

The treatment of the canonical formalism in the present section will focus primarily on the results. Discussion will be held to a minimum, given the significant overlap with the previous example. This is, of course, no surprise considering that the respective propagators are both identical to the Feynman propagator up to multiplication by an appropriate identity matrix.

The propagator $\Delta_{F2}(x)$, as derived in Sec. 4.4, is the Green's function of the Klein-Gordon operator; therefore, the free-field Lagrangian density is given by

$$\mathscr{L}_0(x) = \bar{\psi}(x)\left(\overleftarrow{\partial}^{\mu}\overrightarrow{\partial}_{\mu} - m^2\right)\psi(x), \tag{4.8.1}$$

up to an overall proportionality constant. As will become apparent from what is to follow, the expression here given is consistent in the case of a distinct antiparticle; in the converse scenario, a multiplicative factor of one half is required on the RHS of (4.8.1)

We thus consider the following as canonically conjugate momenta:

$$\Pi(x) \equiv \frac{\partial \mathscr{L}_0(x)}{\partial(\partial_0 \psi(x))} = \partial^0 \bar{\psi}(x), \tag{4.8.2}$$

$$\bar{\Pi}(x) \equiv \frac{\partial \mathscr{L}_0(x)}{\partial(\partial_0 \bar{\psi}(x))} = \partial^0 \psi(x). \tag{4.8.3}$$

Explicitly, in terms of the creation and annihilation operators and the above derived expansion coefficients and dual expansion coefficients, these read

$$\begin{aligned}\Pi_l(x) = \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\left(+ip^0\right)\Big[&\kappa^*\,\mathrm{e}^{+ip\cdot x}\,\bar{u}_l(p;\sigma,s)a^\dagger(p;\sigma,s)\\ &-\lambda^*\,\mathrm{e}^{-ip\cdot x}\,\bar{v}_l(p;\sigma,s)b(p;\sigma,s)\Big],\end{aligned} \tag{4.8.4}$$

and

$$\begin{aligned}\bar{\Pi}_l(x) = \sum_{\sigma,s}\int d^3p\,(2\pi)^{-3/2}\left(-ip^0\right)\Big[&\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p;\sigma,s)a(p;\sigma,s)\\ &-\lambda\,\mathrm{e}^{+ip\cdot x}\,v_l(p;\sigma,s)b^\dagger(p;\sigma,s)\Big].\end{aligned} \tag{4.8.5}$$

4.8.1 Locality

Akin to the treatment in Sec. 3.8.1 of the previous chapter, we must ascertain that certain commutation relations are satisfied before we can claim to have correctly identified a set of canonical field variables. The equal time commutators to be checked are given in (3.8.7)–(3.8.10) to read

$$[\psi_l(\boldsymbol{x},t),\psi_{\bar{l}}(\boldsymbol{y},t)]_- = [\bar{\psi}_l(\boldsymbol{x},t),\bar{\psi}_{\bar{l}}(\boldsymbol{y},t)]_- = [\psi_l(\boldsymbol{x},t),\bar{\psi}_{\bar{l}}(\boldsymbol{y},t)]_- = 0, \tag{4.8.6}$$

$$[\Pi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = [\bar{\Pi}_l(\boldsymbol{x},t),\bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = [\Pi_l(\boldsymbol{x},t),\bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = 0, \tag{4.8.7}$$

$$[\psi_l(\boldsymbol{x},t),\bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = [\bar{\psi}_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = 0, \tag{4.8.8}$$

$$[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = [\bar{\psi}_l(\boldsymbol{x},t),\bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{4.8.9}$$

We now evaluate each of these, first for the case of a distinct antiparticle, and then for that of an indistinct antiparticle.

For a distinct antiparticle

In the present case, in which $a(p;\sigma,s) \neq b(p;\sigma,s)$, many of the above constraints are satisfied solely on account of the commutation relations of the creation and annihilation operators. These include the first two commutators in (4.8.6), the first two commutators in (4.8.7), and the two commutators in (4.8.8). Furthermore, the third commutator in (4.8.6) vanishes as a special case of (4.6.5). As for the third commutator in (4.8.7), this is expanded and evaluated using the transpose of the spin sums given in (4.4.9) and (4.4.10). We thereby obtain

$$[\Pi_l(\boldsymbol{x},t),\bar{\Pi}_{\bar{l}}(\boldsymbol{y},t)]_- = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}p^0p^0\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right], \tag{4.8.10}$$

which manifestly vanishes as required.

For the commutator of the field with its prospective canonically conjugate momentum, we find

$$\begin{aligned}[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\sum_{\sigma,s}\int \frac{d^3p}{(2\pi)^3}p^0\Big[&u_l(p;\sigma,s)\bar{u}_{\bar{l}}(p;\sigma,s)\,\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\\ &+v_l(p;\sigma,s)\bar{v}_{\bar{l}}(p;\sigma,s)\,\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\Big],\end{aligned} \tag{4.8.11}$$

which, upon substitution of the spin sums (4.4.9) and (4.4.10), becomes

$$[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + \mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right].$$

A change of variables, and we obtain

$$[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\int \frac{d^3p}{(2\pi)^3}\,\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = i\,\delta^l{}_{\bar{l}}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{4.8.12}$$

as required by (4.8.9).

For the final commutator, the second in (4.8.9), we expand the LHS to read

$$[\bar\psi_l(\boldsymbol{x},t),\bar\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\sum_{\sigma,s}\int \frac{d^3p}{(2\pi)^3}p^0\left[\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\bar u_l(p;\sigma,s)u_{\bar{l}}(p;\sigma,s)\right.$$
$$\left. +\,\mathrm{e}^{+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\bar v_l(p;\sigma,s)v_{\bar{l}}(p;\sigma,s)\right]. \tag{4.8.13}$$

Noting that $\bar u_l(p;\sigma,s)u_{\bar{l}}(p;\sigma,s) = u_{\bar{l}}(p;\sigma,s)\bar u_l(p;\sigma,s)$, and likewise for $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$, it follows from preceding result that

$$[\bar\psi_l(\boldsymbol{x},t),\bar\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{4.8.14}$$

in agreement with (4.8.9).

For an indistinct antiparticle

Having thus explored the case of a distinct antiparticle, we now consider the converse scenario. We will find that the number of independent canonical fields is here decreased by a factor of one half as compared to the case in which particles and antiparticles are distinct from one another. This is a consequence of the decrease in the number of degrees of freedom that has resulted from the identification of $a(p;\sigma,s)$ with $b(p;\sigma,s)$. To establish the here predicted dependence, we compute the commutators given above in (4.8.8). Expanding the first of these, we obtain

$$[\psi_l(\boldsymbol{x},t),\bar\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = \kappa\lambda i\sum_{\sigma s}\int \frac{d^3p}{(2\pi)^3}p^0\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,u_l(p;\sigma,s)v_{\bar{l}}(p;\sigma,s)\right.$$
$$\left. +\,\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,v_l(p;\sigma,s)u_{\bar{l}}(p;\sigma,s)\right]. \tag{4.8.15}$$

Substituting the spin sums, as evaluated in (4.6.8), this becomes

$$\begin{aligned}[\psi_l(\boldsymbol{x},t),\bar\Pi_{\bar{l}}(\boldsymbol{y},t)]_- &= \kappa\lambda i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\delta^l{}_{\bar{l}}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}+\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right]\\ &= \kappa\lambda\, i\,\delta^l{}_{\bar{l}}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}).\end{aligned} \tag{4.8.16}$$

Similarly, for the second commutator in (4.8.8), we find

$$[\bar{\psi}_l(\boldsymbol{x},t), \Pi_{\bar{l}}(\boldsymbol{y},t)]_- = \kappa^*\lambda^* i \sum_{\sigma s} \int \frac{d^3p}{(2\pi)^3} p^0 \Big[\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\, \bar{u}_l(p;\sigma,s)\bar{v}_{\bar{l}}(p;\sigma,s) + \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\, \bar{v}_l(p;\sigma,s)\bar{u}_{\bar{l}}(p;\sigma,s) \Big]. \quad (4.8.17)$$

It is easy to show, using (4.6.8), that

$$\sum_{\sigma s} \bar{u}_l(p;\sigma,s)\bar{v}_{\bar{l}}(p;\sigma,s) = \sum_{\sigma s} \bar{v}_l(p;\sigma,s)\bar{u}_{\bar{l}}(p;\sigma,s) = \frac{1}{2p^0}\delta^l{}_{\bar{l}}. \quad (4.8.18)$$

Hence, (4.8.17) becomes

$$[\bar{\psi}_l(\boldsymbol{x},t), \Pi_{\bar{l}}(\boldsymbol{y},t)]_- = \kappa^*\lambda^*\, i\, \delta^l{}_{\bar{l}}\, \delta^3(\boldsymbol{x}-\boldsymbol{y}). \quad (4.8.19)$$

We have thus established in (4.8.16), that $\psi(\boldsymbol{x},t)$ and $\bar{\Pi}(\boldsymbol{y},t)$ are not independent; likewise, (4.8.19) demonstrates that $\bar{\psi}(\boldsymbol{x},t)$ and $\Pi(\boldsymbol{y},t)$ fail to be independent. In fact, we can go further with this. Recalling the remark at the end of Sec. 4.7, and in particular the identity given in (4.7.6), which holds under the phase choice (4.6.7), it is easy to show that if we make one further choice of phases, namely $\kappa^* = \lambda$, that is

$$\mathrm{e}^{i(\phi_7+\phi_8)} = 1, \quad (4.8.20)$$

then

$$\bar{\psi}_l(x) = \psi_l(x) \quad \text{and} \quad \bar{\Pi}_l(x) = \Pi_l(x). \quad (4.8.21)$$

Therefore, instead of the four fields considered in the previous section, we here explore whether $\psi(x)$ and $\Pi(x)$ can be interpreted as canonical field variables; that is, we must establish whether the following commutation relations can be satisfied:

$$[\psi_l(\boldsymbol{x},t), \psi_{\bar{l}}(\boldsymbol{y},t)]_- = 0, \quad (4.8.22)$$

$$[\Pi_l(\boldsymbol{x},t), \Pi_{\bar{l}}(\boldsymbol{y},t)]_- = 0, \quad (4.8.23)$$

$$[\psi_l(\boldsymbol{x},t), \Pi_{\bar{l}}(\boldsymbol{y},t)]_- = i\,\delta^l{}_{\bar{l}}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}). \quad (4.8.24)$$

The first of these constraints is met as a special case of (4.6.10). For (4.8.23), we expand the LHS to obtain

$$[\Pi_l(\boldsymbol{x},t), \Pi_{\bar{l}}(\boldsymbol{y},t)]_- = \kappa^*\lambda^* \sum_{\sigma s} \int \frac{d^3p}{(2\pi)^3} p^0 p^0 \Big[-\,\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\, \bar{u}_l(p;\sigma,s)\bar{v}_{\bar{l}}(p;\sigma,s) + \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\, \bar{v}_l(p;\sigma,s)\bar{u}_{\bar{l}}(p;\sigma,s) \Big]. \quad (4.8.25)$$

From (4.8.18), this is simply

$$[\Pi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_- = \kappa^*\lambda^*\delta^l{}_{\bar{l}}\int\frac{d^3p}{(2\pi)^3}\frac{p^0}{2}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}-\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right], \tag{4.8.26}$$

and thus vanishes as desired. For the final commutator we note by inspection of $\psi(x)$ and $\Pi(x)$, as given above in (4.1.1) and (4.8.4) respectively, that $[\psi_l(\boldsymbol{x},t),\Pi_{\bar{l}}(\boldsymbol{y},t)]_-$, when expanded, is identical to (4.8.11). We thus conclude from (4.8.12) that (4.8.24) is satisfied.

This completes the analysis of the locality structure of the present theory. In the next section we develop the Hamiltonian formalism in the case of a distinct antiparticle. The converse case is not explicitly considered because this follows trivially from the here to be derived results.

4.8.2 The Hamiltonian

According to the standard prescription $\mathcal{H}_0$ is given by

$$\mathcal{H}_0 = \int d^3x\mathscr{H}_0(x), \tag{4.8.27}$$

where

$$\mathscr{H}_0(x) = \Pi(x)\partial_0\psi(x) + \bar{\Pi}(x)\partial_0\bar{\psi}(x) - \mathscr{L}_0(x). \tag{4.8.28}$$

Inserting (4.8.2), (4.8.3), and (4.8.1), we obtain

$$\begin{aligned}\mathscr{H}_0(x) &= \partial^0\bar{\psi}(x)\partial_0\psi(x) + \partial^0\psi(x)\partial_0\bar{\psi}(x) - \partial^\mu\bar{\psi}(x)\partial_\mu\psi(x) + m^2\bar{\psi}(x)\psi(x)\\ &= \partial^0\psi(x)\partial_0\bar{\psi}(x) + \boldsymbol{\nabla}\bar{\psi}(x)\cdot\boldsymbol{\nabla}\psi(x) + m^2\bar{\psi}(x)\psi(x).\end{aligned} \tag{4.8.29}$$

We now show that (4.8.27) reduces to the free-particle Hamiltonian. Given that the formal expression (4.8.29) is identical to (3.8.35) we shall refrain from repeating the explicit expansion in terms of the underlying creation and annihilation operators. In order to infer the expansion of (4.8.29) using the results of Sec. 3.8.2 we must check the orthogonality properties of the coefficient functions. These are readily computed; we obtain

$$\bar{\boldsymbol{u}}(\boldsymbol{p};\sigma,s)\boldsymbol{u}(\boldsymbol{p};\sigma',s') = \frac{1}{2p^0}\delta_{\sigma\sigma'}\delta_{ss'}, \tag{4.8.30}$$

$$\bar{\boldsymbol{v}}(\boldsymbol{p};\sigma,s)\boldsymbol{v}(\boldsymbol{p};\sigma',s') = \frac{1}{2p^0}\delta_{\sigma\sigma'}\delta_{ss'}. \tag{4.8.31}$$

Comparison with (3.8.36) and (3.8.37) confirms that the results of Sec. 3.8.2 can be directly applied in the present context. We thus obtain

$$\begin{aligned}\mathcal{H}_0 = \frac{1}{2}\sum_{\sigma,s}\int d^3p\, p^0\Big[&a^\dagger(\boldsymbol{p};\sigma,s)a(\boldsymbol{p};\sigma,s) + b(\boldsymbol{p};\sigma,s)b^\dagger(\boldsymbol{p};\sigma,s)\\ &+a(\boldsymbol{p};\sigma,s)a^\dagger(\boldsymbol{p};\sigma,s) + b^\dagger(\boldsymbol{p};\sigma,s)b(\boldsymbol{p};\sigma,s)\Big].\end{aligned} \tag{4.8.32}$$

Using the commutation relations of the creation and annihilation operators, as per (2.4.3) and (2.4.4), $\mathcal{H}_0$ in normal ordered form becomes

$$\mathcal{H}_0 = \sum_{\sigma,s} \int d^3p \, p^0 \left[a^\dagger(\boldsymbol{p};\sigma,s) a(\boldsymbol{p};\sigma,s) + b^\dagger(\boldsymbol{p};\sigma,s) b(\boldsymbol{p};\sigma,s) + \delta(\boldsymbol{p}-\boldsymbol{p}) \right] . \tag{4.8.33}$$

Looking at Weinberg's general treatment in [40, p. 296] confirms that (4.8.33) is the desired result; namely, the free-particle Hamiltonian plus an infinite constant term.

5
Conclusion

We have developed an extension of the derivation of quantum fields by Weinberg whereby it is possible to construct quantum fields with more than one spin degree of freedom. Specifically, we have shown that massive bosonic quantum fields of the type $(j/2, j/2)$ can be constructed so as to include $j+1$ spin degrees of freedom: $s \in \{j, j-1, \ldots, 0\}$. Such quantum fields allow for a spin-dependent metric. We have demonstrated in two particular cases, $j=1$ and $j=2$, that the spin-dependence of the metric can be chosen such that the ensuing quantum field theory admits the relevant Feynman spin j propagator and is therefore consistent and unitary at all energies without the need for regulator terms. The here developed formalism is sufficiently general so as to allow for the construction of massive bosonic quantum fields that include fewer than $j+1$ spin degrees of freedom. That is to say, one may construct a quantum field with highest spin degree of freedom j and include any subset of the lower spin degrees of freedom from the set $\{j-1, \ldots, 0\}$. For instance, in the case of the $(1/2, 1/2)$ quantum field of Ch. 3, we could have chosen to include only $s=0$ or only $s=1$ or, as explicitly demonstrated, both $s=0$ and $s=1$. Notwithstanding, the here obtained result, of a theory that has a highest spin degree of freedom j and that is consistent and unitary at all energies without the need for regulator terms, appears to be reliant upon the appropriate use of all $j+1$ spin degrees of freedom. This is certainly true in the two examples here considered and it is likely to hold in general on account of the arguments offered in favour of the conjecture of Sec. 2.7.

An outstanding question is that of a phenomenological application of the particular field theories constructed in Chs. 3 and 4. In the case of a theory with spin one and spin zero degrees of freedom, there is an existing body of literature [62, 109, 113–117] on possible phenomenological models for the description of weak interactions. No such counterpart is known to the author in the case of the field theory of Ch. 4, although there are numerous publications on massive gravity. This may prove to be a worthy avenue for further exploration.

6

Massless quantum fields

The present chapter on massless fields is included as an aside. We begin by a brief review of the general formalism provided by Weinberg in [40] for the derivation of massless fields of type (j, j'). We then consider the specific case of a spin one-half particle. Here we show that a field of the representation $(1/2, 0)$ can destroy particles of helicity $\sigma = +1/2$ and create antiparticles of helicity $\sigma = -1/2$. On the other hand, a field of the representation $(1/2, 0) \oplus (0, 1/2)$ can destroy particles of helicity $\sigma = \pm 1/2$ and create antiparticles of helicity $\sigma = \pm 1/2$.

6.1 The quantum field

The abstract expression for the quantum field of a massless particle [40, p. 247] is no different to that of a massive particle as given in Sec. 2.5. We thus have

$$\psi_l(x) = \sum_\sigma \int d^3p\,(2\pi)^{-3/2}\left[\kappa\,\mathrm{e}^{-ip\cdot x}\,u_l(p,\sigma)a(p,\sigma) + \lambda\,\mathrm{e}^{ip\cdot x}\,v_l(p,\sigma)b^\dagger(p,\sigma)\right], \tag{6.1.1}$$

where $p^0 = |\boldsymbol{p}|$ and $\kappa, \lambda \in \mathbb{C}$. The coefficient functions are chosen such that the field will transform under the action of the unitary representations of $\mathcal{L}_+^\uparrow$ according to

$$U[\Lambda]\psi_l(x)U^{-1}[\Lambda] = \sum_{\bar{l}} D_{l\bar{l}}[\Lambda^{-1}]\psi_{\bar{l}}(\Lambda x), \tag{6.1.2}$$

where the position independent matrix $D[\Lambda^{-1}]$ furnishes a representation of $\mathcal{L}_+^\uparrow$ on the space of coefficient functions. Akin to the definition given in Sec. 2.4, $a(p,\sigma)$ and $b^\dagger(p,\sigma)$ are particle annihilation and antiparticle creation operators, respectively. In the present context $a(p,\sigma)$ destroys a particle of momentum p and helicity σ; $b^\dagger(p,\sigma)$ creates an antiparticle with momentum p and helicity σ. Recalling the transformation properties of the state vectors (2.3.40), we immediately obtain the transformation properties of the creation and annihilation operators to read

$$U[\Lambda]a(p,\sigma)U^{-1}[\Lambda] = \sqrt{\frac{(\Lambda p)^0}{p^0}}\, \mathrm{e}^{-i\sigma\theta(\Lambda,p)}\, a(\Lambda p,\sigma), \tag{6.1.3}$$

$$U[\Lambda]b^\dagger(p,\sigma)U^{-1}[\Lambda] = \sqrt{\frac{(\Lambda p)^0}{p^0}}\, \mathrm{e}^{i\sigma\theta(\Lambda,p)}\, b^\dagger(\Lambda p,\sigma). \tag{6.1.4}$$

These transformation properties, taken together with (6.1.2), yield the following constraints on the $\boldsymbol{u}(p,\sigma)$ and $\boldsymbol{v}(p,\sigma)$ coefficient functions:

$$u_l(\Lambda p,\sigma)\, \mathrm{e}^{i\sigma\theta(p,\Lambda)} = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\bar{l}} D_{l\bar{l}}[\Lambda] u_{\bar{l}}(p,\sigma), \tag{6.1.5}$$

$$v_l(\Lambda p,\sigma)\, \mathrm{e}^{-i\sigma\theta(p,\Lambda)} = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\bar{l}} D_{l\bar{l}}[\Lambda] v_{\bar{l}}(p,\sigma), \tag{6.1.6}$$

where $\theta(\Lambda,p)$ is the angle in (2.3.40). In the basis in which the standard vector is given by $k=(\kappa,0,0,\kappa)$, this angle is an angle of rotation about the z-axis.

The constraints (6.1.5) and (6.1.6) can be put into a more useful form by considering particular Lorentz transformations in turn, as we did for massive fields in Sec. 2.5.3. We begin by considering $p=k$ and $\Lambda=W$, where W is an element of the little group for massless particles. In Sec. 2.1.1, this was found to be $\mathrm{ISO}(2)$, the group of translations and rotations in two dimension. The constraints thus become

$$u_l(k,\sigma)\, \mathrm{e}^{i\sigma\theta(k,W)} = \sum_{\bar{l}} D_{l\bar{l}}[W] u_{\bar{l}}(k,\sigma), \tag{6.1.7}$$

$$v_l(k,\sigma)\, \mathrm{e}^{-i\sigma\theta(k,W)} = \sum_{\bar{l}} D_{l\bar{l}}[W] v_{\bar{l}}(k,\sigma). \tag{6.1.8}$$

Noting that $D[W]$ furnishes a representation of $\mathrm{ISO}(2)$ on the space of coefficient functions, we can expand the little group transformations on both sides of (6.1.7) and (6.1.8) to first order in terms of the underlying infinitesimal generators. We thus obtain

$$u_l(k,\sigma)\,\{1+i\sigma\theta\} = \sum_{\bar{l}} \{\mathbb{1} + i\alpha A + i\beta B + i\theta \mathscr{J}_z\}_{l\bar{l}}\, u_{\bar{l}}(k,\sigma), \tag{6.1.9}$$

$$v_l(k,\sigma)\,\{1-i\sigma\theta\} = \sum_{\bar{l}} \{\mathbb{1} + i\alpha A + i\beta B + i\theta \mathscr{J}_z\}_{l\bar{l}}\, v_{\bar{l}}(k,\sigma), \tag{6.1.10}$$

where $\mathbb{1}$ is an identity matrix of the appropriate dimensions and, from Sec. 2.1.1, $A = \mathscr{K}_x + \mathscr{J}_y$ and $B = \mathscr{K}_y - \mathscr{J}_x$. Equating coefficients on both sides of (6.1.9) and (6.1.10), and expressing the resultant constraints free of row index l, we have

$$A\boldsymbol{u}(k,\sigma) = B\boldsymbol{u}(k,\sigma) = 0, \quad J_z\boldsymbol{u}(k,\sigma) = +\sigma\,\boldsymbol{u}(k,\sigma), \tag{6.1.11}$$

$$A\boldsymbol{v}(k,\sigma) = B\boldsymbol{v}(k,\sigma) = 0, \quad J_z\boldsymbol{v}(k,\sigma) = -\sigma\,\boldsymbol{v}(k,\sigma). \tag{6.1.12}$$

These constraints will allow us to determine the coefficient functions at standard momentum k in terms of a finite number of free parameters.

In order to obtain the coefficient functions at momentum $q = (|\boldsymbol{q}|, \boldsymbol{q})$, consider (6.1.5) and (6.1.6) with $p = k$ and $\Lambda = L(q)$, where $L(q)$ is the standard Lorentz boost defined by $q^\mu = L^\mu{}_\nu(q)k^\nu$, as in (2.1.9). Constraints (6.1.5) and (6.1.6) then read

$$u_l(q,\sigma) = \sqrt{\frac{\kappa}{|\boldsymbol{q}|}}\sum_{\bar{l}} D_{l\bar{l}}[L(q)]u_{\bar{l}}(k,\sigma), \tag{6.1.13}$$

$$v_l(q,\sigma) = \sqrt{\frac{\kappa}{|\boldsymbol{q}|}}\sum_{\bar{l}} D_{l\bar{l}}[L(q)]v_{\bar{l}}(k,\sigma). \tag{6.1.14}$$

The coefficient functions at $q = (|\boldsymbol{q}|, \boldsymbol{q})$ are thus related to those at $k = (\kappa, 0, 0, \kappa)$, up to proportionality, by $D[L(q)]$, a representation of the standard boost for a massless particle of positive energy. Looking at the treatment in App. B.2.2, we find that it is most convenient to express $L(p)$ in terms of the following succession of Lorentz transformations:

$$L(p) = R(\hat{\boldsymbol{p}})B(|\boldsymbol{p}|), \tag{6.1.15}$$

where $B(|\boldsymbol{p}|)$ is a boost operator that maps $(\kappa, 0, 0, \kappa)$ to $(|\boldsymbol{p}|, 0, 0, |\boldsymbol{p}|)$, and $R(\hat{\boldsymbol{p}})$ is the operator that rotates $(0, 0, |\boldsymbol{p}|)$ into the direction $\hat{\boldsymbol{p}}$. If $\boldsymbol{p}$ is expressed in spherical coordinates

$$p_x = |\boldsymbol{p}|\cos(\theta)\sin(\phi), \quad p_y = |\boldsymbol{p}|\sin(\theta)\sin(\phi), \quad \text{and} \quad p_z = |\boldsymbol{p}|\cos(\phi), \tag{6.1.16}$$

where $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$ are the polar and azimuthal angles, respectively, then

$$R(\hat{\boldsymbol{p}}) = R(\theta)R(\phi), \tag{6.1.17}$$

where $R(\phi)$ is a rotation about the y-axis by an angle negative ϕ; $R(\theta)$ is a rotation about the z-axis by an angle negative θ. Consequently, $D[L(p)]$ is given by

$$D[L(p)] = D[R(\theta)]D[R(\phi)]D[B(|\boldsymbol{p}|)]. \tag{6.1.18}$$

This expansion will prove useful toward the derivation of explicit representations of the standard boost in Secs. 6.2 and 6.3.

Through the derivation of the above constraints, we have reduced the task of deriving quantum fields for massless particles of any desired helicity to the task of solving (6.1.11) and (6.1.12) to obtain the coefficient functions. The question remains: how should one choose the representation $D[\Lambda]$ so as to obtain a field that will create and destroy particles of a given desired helicity? In [40, p. 254], Weinberg shows that there exist non-trivial solutions of (6.1.11) and (6.1.12) for the description of particles of helicity σ and antiparticles of helicity $-\sigma$ only if the representation $D[\Lambda]$ of (j, j') is chosen such that[1]

$$\sigma = j - j'. \tag{6.1.19}$$

This will be exemplified in the next two sections as we consider the representations $(1/2, 0)$ and $(1/2, 0) \oplus (0, 1/2)$, respectively.

6.2 The $(1/2, 0)$ representation

We here derive the coefficients functions for a field $\psi(x)$ that transforms under the unitary representations of the restricted Lorentz group according to (6.1.2) where the representation $D[\Lambda]$ is chosen to be $(1/2, 0)$.

The $(1/2, 0)$ representation of the restricted Lorentz group is defined by the generators

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{K} = \frac{1}{2i}\boldsymbol{\sigma}, \tag{6.2.1}$$

where $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$. The Pauli matrices read

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.2.2}$$

Hence, the three generators of the little group are given by

$$A = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.2.3}$$

It apparent by inspection of the constraints (6.1.11) and (6.1.12) that the coefficient functions at momentum k must be of the form

$$\boldsymbol{u}(k, +1/2) = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}(k, -1/2) = \begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix}, \tag{6.2.4}$$

where $\alpha_i \in \mathbb{C}$. The requirement that $\boldsymbol{u}(k, \sigma)$ and $\boldsymbol{v}(k, \sigma)$ be annihilated by the A and B given in (6.2.3) precludes the possibility of a non-zero entry in the lower component. The third constraint in (6.1.11) and (6.1.12) determines the helicity label of the u-coefficient to be $\sigma = +1/2$ and that of the v-coefficient to be $\sigma = -1/2$. This confirms that the field here constructed from the $(1/2, 0)$ representation will destroy a particle of helicity $+1/2$ and create an antiparticle of helicity $-1/2$, in accordance with the general result of Weinberg cited above in (6.1.19).

Now, to find the coefficient functions at momentum p, we must construct the boost operator $D[L(p)]$ for the $(1/2, 0)$ representation. This is achieved in the usual fashion via

[1] The expression given by Weinberg in [40, p. 254] differs by an overall minus sign because of the sign conventions chosen in the definitions in [40, p. 230].

the exponential map.

We begin by performing a Maclaurin series expansion of the exponential $\exp(i\boldsymbol{K}\cdot\boldsymbol{\varphi})$, where $\boldsymbol{K}$ represents the boost generators given in (6.2.1) and $\boldsymbol{\varphi}$ is the rapidity vector defined in terms of the three-momentum such that

$$\boldsymbol{\varphi} = \varphi\,\hat{\boldsymbol{p}}, \quad \text{with} \quad \varphi \equiv \log\left(\frac{|\boldsymbol{p}|}{\kappa}\right). \tag{6.2.5}$$

From the result obtained in (C.3.6), we have

$$\exp(i\boldsymbol{K}\cdot\boldsymbol{\varphi}) = \mathbb{1}_2\cosh(\varphi/2) + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\sinh(\varphi/2). \tag{6.2.6}$$

Invoking the relevant half angle formulae for a real argument

$$\sinh(x/2) = \operatorname{sgn}(x)\sqrt{\frac{\cosh(x)-1}{2}} \quad \text{and} \quad \cosh(x/2) = \sqrt{\frac{\cosh(x)+1}{2}}, \tag{6.2.7}$$

and noting from (6.2.5) that the rapidity parameter φ is positive definite, we obtain

$$\exp(i\boldsymbol{K}\cdot\boldsymbol{\varphi}) = \mathbb{1}_2\sqrt{\frac{\cosh(\varphi)+1}{2}} + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\sqrt{\frac{\cosh(\varphi)-1}{2}}. \tag{6.2.8}$$

Hence, inserting φ from (6.2.5) into (6.2.8), we obtain

$$\begin{aligned}
\exp(i\boldsymbol{K}\cdot\boldsymbol{\varphi}) &= \mathbb{1}_2\sqrt{\frac{\cosh(\log(|\boldsymbol{p}|/\kappa))+1}{2}} + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\sqrt{\frac{\cosh(\log(|\boldsymbol{p}|/\kappa))-1}{2}} \\
&= \mathbb{1}_2\frac{|\boldsymbol{p}|+\kappa}{2\sqrt{\kappa|\boldsymbol{p}|}} + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\frac{|\boldsymbol{p}|-\kappa}{2\sqrt{\kappa|\boldsymbol{p}|}} \\
&= \frac{|\boldsymbol{p}|+\kappa}{2\sqrt{\kappa|\boldsymbol{p}|}}\left[\mathbb{1}_2 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\frac{|\boldsymbol{p}|-\kappa}{|\boldsymbol{p}|+\kappa}\right].
\end{aligned} \tag{6.2.9}$$

The boost $D[B(|\boldsymbol{p}|)]$ is readily obtained by evaluating (6.2.9) with $\hat{\boldsymbol{p}} = (0,0,1)$:

$$D[B(|\boldsymbol{p}|)] = \frac{|\boldsymbol{p}|+\kappa}{2\sqrt{\kappa|\boldsymbol{p}|}}\left[\mathbb{1}_2 + \sigma_z\frac{|\boldsymbol{p}|-\kappa}{|\boldsymbol{p}|+\kappa}\right]. \tag{6.2.10}$$

Turning now to the rotations, we perform a Maclaurin series expansion of the exponential $\exp(i\boldsymbol{J}\cdot\boldsymbol{\theta})$; here, $\boldsymbol{J}$ represents the rotation generators given above in (6.2.1); the associated parameters are $\boldsymbol{\theta} \equiv (\theta_x, \theta_y, \theta_z)$. From the result obtained in (C.3.3), we have

$$\exp(i\boldsymbol{J}\cdot\boldsymbol{\theta}) = \mathbb{1}_2\cos(\theta/2) + i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\theta}}\sin(\theta/2). \tag{6.2.11}$$

Hence, the desired rotation operators read

$$D[R(\phi)] = \mathbb{1}_2 \cos(\phi/2) - i\sigma_y \sin(\phi/2), \tag{6.2.12}$$

$$D[R(\theta)] = \mathbb{1}_2 \cos(\theta/2) - i\sigma_z \sin(\theta/2). \tag{6.2.13}$$

The coefficient functions are thus given at momentum p by

$$u_l(p, +1/2) = \sqrt{\frac{\kappa}{|\boldsymbol{p}|}} \sum_{\bar{l}} D_{l\bar{l}}[L(p)] u_{\bar{l}}(k, +1/2), \tag{6.2.14}$$

$$v_l(p, -1/2) = \sqrt{\frac{\kappa}{|\boldsymbol{p}|}} \sum_{\bar{l}} D_{l\bar{l}}[L(p)] v_{\bar{l}}(k, -1/2), \tag{6.2.15}$$

where $\boldsymbol{u}(k, +1/2)$ and $\boldsymbol{v}(k, -1/2)$ are as given in (6.2.4) and $D[L(p)]$ is given by (6.1.18) with constituent transformations operators $D[B(|\boldsymbol{p}|)]$, $D[R(\phi)]$, and $D[R(\theta)]$ as derived in (6.2.10), (6.2.12), and (6.2.13), respectively.

6.3 The $(1/2, 0) \oplus (0, 1/2)$ representation

We now derive the coefficients functions for a field $\psi(x)$ that transforms under the unitary representations of the restricted Lorentz group according to (6.1.2) where the representation $D[\Lambda]$ is chosen to be $(1/2, 0) \oplus (0, 1/2)$.

The generators of the Lie algebra of the $(1/2, 0) \oplus (0, 1/2)$ representation of the restricted Lorentz group read

$$\boldsymbol{J} \equiv \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & \mathbb{O}_2 \\ \mathbb{O}_2 & \boldsymbol{\sigma} \end{pmatrix} \quad \text{and} \quad \boldsymbol{K} \equiv \frac{1}{2i}\begin{pmatrix} \boldsymbol{\sigma} & \mathbb{O}_2 \\ \mathbb{O}_2 & -\boldsymbol{\sigma} \end{pmatrix}, \tag{6.3.1}$$

where, as above, $\boldsymbol{\sigma}$ are the three Pauli matrices and $\mathbb{O}_2$ is a 2×2 zero matrix. Accordingly, the generators of the little group read

$$A = \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & +i & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; J_z = \frac{1}{2}\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The constraint that the coefficient functions must be annihilated by A and B immediately tells us that $\boldsymbol{u}(k, \sigma)$ and $\boldsymbol{v}(k, \sigma)$ may have non-zero entries only in the top most and bottom most components. This, along with the requirement that $\boldsymbol{u}(k, \sigma)$ and $\boldsymbol{v}(k, \sigma)$ must have eigenvalues of $+\sigma$ and $-\sigma$, respectively, under the action of J_z, gives

$$\boldsymbol{u}(k,+1/2)=\begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix},\qquad \boldsymbol{u}(k,-1/2)=\begin{pmatrix}0\\0\\0\\\alpha_2\end{pmatrix},\tag{6.3.2}$$

$$\boldsymbol{v}(k,+1/2)=\begin{pmatrix}0\\0\\0\\\alpha_3\end{pmatrix},\qquad \boldsymbol{v}(k,-1/2)=\begin{pmatrix}\alpha_4\\0\\0\\0\end{pmatrix},\tag{6.3.3}$$

where $\alpha_i\in\mathbb{C}$ are free parameters that may be determined using the constraints on the (anti-) commutation relations of the field with itself and its adjoint, as motivated in Sec. 2.5. We shall not perform this exercise in the present work.

Thanks to the block diagonal structure of the generators (6.3.1), the standard boost for the $(1/2,0)\oplus(0,1/2)$ representation can be easily deduced from the results of the previous section. Using (6.2.10), the matrix that induces a boost from κ to $|\boldsymbol{p}|$ along the z-axis is given by

$$D[B(|\boldsymbol{p}|)]=\frac{|\boldsymbol{p}|+\kappa}{2\sqrt{\kappa|\boldsymbol{p}|}}\begin{pmatrix}\mathbb{1}_2+\sigma_z\frac{|\boldsymbol{p}|-\kappa}{|\boldsymbol{p}|+\kappa} & \mathbb{O}_2\\ \mathbb{O}_2 & \mathbb{1}_2-\sigma_z\frac{|\boldsymbol{p}|-\kappa}{|\boldsymbol{p}|+\kappa}\end{pmatrix}.\tag{6.3.4}$$

Furthermore, from (6.2.12), we have

$$D[R(\phi)]=\begin{pmatrix}\mathbb{1}_2\cos(\phi/2)-i\sigma_y\sin(\phi/2) & \mathbb{O}_2\\ \mathbb{O}_2 & \mathbb{1}_2\cos(\phi/2)-i\sigma_y\sin(\phi/2)\end{pmatrix}.\tag{6.3.5}$$

Similarly, from (6.2.13), we have

$$D[R(\theta)]=\begin{pmatrix}\mathbb{1}_2\cos(\theta/2)-i\sigma_z\sin(\theta/2) & \mathbb{O}_2\\ \mathbb{O}_2 & \mathbb{1}_2\cos(\theta/2)-i\sigma_z\sin(\theta/2)\end{pmatrix}.\tag{6.3.6}$$

The coefficient functions at momentum p are then given by

$$u_l(p,\sigma)=\sqrt{\frac{\kappa}{|\boldsymbol{p}|}}\sum_{\bar{l}}D_{l\bar{l}}[L(p)]u_{\bar{l}}(k,\sigma),\tag{6.3.7}$$

$$v_l(p,\sigma)=\sqrt{\frac{\kappa}{|\boldsymbol{p}|}}\sum_{\bar{l}}D_{l\bar{l}}[L(p)]v_{\bar{l}}(k,\sigma),\tag{6.3.8}$$

where $\boldsymbol{u}(k,\sigma)$ and $\boldsymbol{v}(k,\sigma)$ are given in (6.3.2) and (6.3.3), respectively; $D[L(p)]$ is composed, as in (6.1.18), from constituent transformations operators $D[B(|\boldsymbol{p}|)]$, $D[R(\phi)]$, and $D[R(\theta)]$ given by (6.3.4), (6.3.5), and (6.3.6), respectively.

6.4 Conclusion

We have given a brief review of Weinberg's prescription for the derivation of quantum fields for a massless particle of positive energy and provided explicit examples thereof by constructing fields of the $(1/2,0)$ and $(1/2,0)\oplus(0,1/2)$ representations, respectively. In Sec. 6.2 we found that a field of the representation $(1/2,0)$ destroys particles of helicity $+1/2$ and creates antiparticles of helicity $-1/2$. In Sec. 6.3 found that a field of the representation $(1/2,0)\oplus(0,1/2)$ destroys particles of helicity $\pm1/2$ and creates antiparticles of helicity $\pm1/2$. As one might expect, the coefficient functions derived in (6.3.7) and (6.3.8) agree with the massless limit of the massive Dirac coefficient functions given by Weinberg in [40, Sec. 5.5].

7

Derivation of free-space Proca and Maxwell equations

We here provide a derivation of the free-space Proca and Maxwell equations by considering the transformation properties of a six-component classical field under the $(1,0) \oplus (0,1)$ representation of the orthochronous Lorentz group. The orthochronous Lorentz group is given by the restricted Lorentz group together with space-inversion and is denoted by $\mathcal{L}^{\uparrow}$ [98, p. 11]. Before we proceed, a brief review of earlier attempts will be in order.

A search of the literature suggests that it is Weinberg who first employed a spin one representation of the Lorentz group toward the derivation of Maxwell's equations. In [65, 66] and [118, p. 405], Weinberg constructs quantum fields for massless particles of positive energy. In the particular case of the representations $(1,0)$ and $(0,1)$, he shows that the linear combinations of the corresponding particle annihilation (creation) fields and antiparticle creation (annihilation) fields satisfy a first order differential equation. The Maxwell equations for electric and magnetic fields in free-space are then obtained via a suitably chosen identification of the components of the free fields with the six components of the electromagnetic field strength tensor.

Subsequent work by Ahluwalia [119] pioneered a new approach toward a derivation of Maxwell's equations from the massless limit of the second order field equations of six-component spinors of the $(1,0) \oplus (0,1)$ representation of $\mathcal{L}^{\uparrow}$. Contrary to a theorem of Weinberg [64], whereby there exists a well defined massless limit for all fields of the form $(j,0) \oplus (0,j)$, it was reported [119–121] that the equations derived in the case $m = 0$ fail to coincide with those obtained under the $m \to 0$ limit. In particular, a direct identification of the linear combinations of the $(1,0)$ and $(0,1)$ spinors with the six components of the electromagnetic field strength tensor, was shown to yield the six dynamical Maxwell equations, albeit with an extra curl. The two constraints for electric and magnetic fields in free-space were not obtained. Furthermore, it was shown that the six equations with the extra curl can be rewritten in the form of Maxwell's equations with intrinsic source terms, where these source terms arise as the gradients of two scalar fields [122]. Notwithstanding, this result was deemed favourable insofar as it does not suffer from the so called "kinematic acausality" that reportedly plagued Weinberg's spin one equations for massless particles [119, 120, 122–126].

The approach taken by Ahluwalia can be considered a direct spin one counterpart of Ryder's derivation of the Dirac equation in [82, Ch. 2]. We here follow this approach in the first part of our derivation to obtain the second order field equation for a six-component classical field of the representation $(1,0)\oplus(0,1)$ of $\mathcal{L}^{\uparrow}$. We then show that the above difficulties can be ameliorated via an alternate identification: instead of identifying electric and magnetic fields directly with the linear combinations of the $(1,0)$ and $(0,1)$ fields, one ought to identify electric and magnetic fields with the curls of the linear combinations of the $(1,0)$ and $(0,1)$ fields. The here proposed identification yields the Proca equations and, in the $m\to 0$ limit, a complete set of free-space Maxwell equations.

7.1 The $(1,0)\oplus(0,1)$ representation of the orthochronous Lorentz group

In Sec. 2.2, we derived the Lie algebra of the restricted Poincaré group. The subalgebra defined by the first three commutation relations, (2.2.22)–(2.2.24), reads

$$[J_i, J_j] = -i\epsilon_{ijk}J_k, \tag{7.1.1}$$

$$[J_i, K_j] = -i\epsilon_{ijk}K_k, \tag{7.1.2}$$

$$[K_i, K_j] = +i\epsilon_{ijk}J_k. \tag{7.1.3}$$

This is, of course, the Lie algebra of the restricted Lorentz group where J_i and K_i, $i\in\{x,y,x\}$, are the generators of rotations and boosts, respectively. The completely antisymmetric Levi-Civita symbol is defined by $\epsilon_{xyz}=-1$. The standard spin one angular momentum matrices can be constructed using the relations derived in (2.3.30). We thereby obtain

$$J_x=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad J_y=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix},\quad J_z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}. \tag{7.1.4}$$

Instead of this representation it will prove algebraically favourable to use the so called adjoint representation.[1] Using $\mathcal{J}_i$, $i\in\{x,y,z\}$, to denote the rotation generators in the adjoint representation we find that these matrices are related to the above via a similarity transformation, $\mathcal{J}_i = SJ_iS^{-1}$, where S is the unitary matrix

$$S=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&-1\\i&0&i\\0&-\sqrt{2}&0\end{pmatrix}. \tag{7.1.5}$$

The rotation generators in the adjoint representation thus read

$$\mathcal{J}_x=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \mathcal{J}_y=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},\quad \mathcal{J}_z=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}. \tag{7.1.6}$$

[1] The adjoint representation is obtained from the structure constants of the Lie algebra. In the case at hand, these are simply $i\epsilon_{ijk}$. Being completely antisymmetric, this choice will prove convenient later as it will allow us to rewrite $\boldsymbol{J}\cdot\boldsymbol{\nabla}$ in terms of a curl.

It is easy to show that these matrices, along with boost generators given by $\boldsymbol{K} = \pm i\boldsymbol{J}$, where $\boldsymbol{J} \equiv (\mathcal{J}_x, \mathcal{J}_y, \mathcal{J}_z)$, satisfy the above Lie algebra. We may thus introduce two irreducible spin one representations of the restricted Lorentz group: $(1,0)$ given by the generators $\{\boldsymbol{J}, -i\boldsymbol{J}\}$, and $(0,1)$ given by $\{\boldsymbol{J}, +i\boldsymbol{J}\}$. Considering that angular momentum is quadratic in spatial coordinates, and momentum is linear, the symmetry of space-inversion, $\{\boldsymbol{x} \to -\boldsymbol{x},\ \boldsymbol{p} \to -\boldsymbol{p}\}$, will map $(1,0) \leftrightarrow (0,1)$. Hence, if we are to construct a parity covariant theory, we must consider the direct sum representation $(1,0)\oplus(0,1)$. The generators of this representation are given by the direct sum of the generators of $(1,0)$ and $(0,1)$. The corresponding operators will be derived in the next section by exponentiation. It directly follows that the space-inversion operator in the $(1,0) \oplus (0,1)$ representation is given by

$$\beta \equiv \begin{pmatrix} \mathbb{O}_3 & \mathbb{1}_3 \\ \mathbb{1}_3 & \mathbb{O}_3 \end{pmatrix}. \tag{7.1.7}$$

Here, $\mathbb{O}_n$ is an $n \times n$ null matrix; $\mathbb{1}_n$ is $n \times n$ identity matrix. This expression will be verified in the next section.

7.2 The wave equation

We now derive the relativistic spin one wave equation as an identity among the elements of a six-component classical field of the $(1,0) \oplus (0,1)$ representation of $\mathcal{L}^{\uparrow}$. We begin by introducing two three-component classical fields $\boldsymbol{X} \equiv (X_i)$ and $\boldsymbol{Y} \equiv (Y_i)$, $i \in \{1,2,3\}$, defined by their respective transformation properties under Lorentz boost in following manner:

$$(1,0): \qquad \boldsymbol{X}(\boldsymbol{p}) = \exp\left[+\boldsymbol{J}\cdot\boldsymbol{\varphi}\right]\boldsymbol{X}(0), \tag{7.2.1}$$

$$(0,1): \qquad \boldsymbol{Y}(\boldsymbol{p}) = \exp\left[-\boldsymbol{J}\cdot\boldsymbol{\varphi}\right]\boldsymbol{Y}(0), \tag{7.2.2}$$

where $\boldsymbol{\varphi} \equiv \varphi\,\hat{\boldsymbol{p}}$ is the rapidity vector defined in terms of its magnitude, the rapidity parameter φ, and its direction, the unit three-momentum $\hat{\boldsymbol{p}} \equiv \boldsymbol{p}/\left|\boldsymbol{p}\right|$. The rapidity parameter is given by the relations

$$m\cosh(\varphi) = E = \sqrt{\left|\boldsymbol{p}\right|^2 + m^2} \quad \text{and} \quad m\sinh(\varphi) = \left|\boldsymbol{p}\right|. \tag{7.2.3}$$

The fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ are chosen to be equal at rest:

$$\boldsymbol{X}(0) = \boldsymbol{Y}(0). \tag{7.2.4}$$

In his derivation of the Dirac equation Ryder motivates this choice by giving a physical interpretation [82, p. 41] to his spin one-half counterparts of $\boldsymbol{X}(0)$ and $\boldsymbol{Y}(0)$, the Weyl spinors. We shall not take any such liberties here. Ahluwalia [121] points out that $\boldsymbol{X}(0)$ and $\boldsymbol{Y}(0)$ may differ by a sign. We will not explicitly consider this possibility here for the sake of simplicity. We will, however, remark on the consequences of the choice $\boldsymbol{X}(0) = \boldsymbol{Y}(0)$ at various stages throughout in the subsequent development.

Having thus defined the three-component fields of the $(1,0)$ and $(0,1)$ representations, respectively, we may now introduce a six-component field $\boldsymbol{\Psi}$ as follows:

$$\boldsymbol{\Psi}(0) \equiv \begin{pmatrix} \boldsymbol{X}(0) \\ \boldsymbol{Y}(0) \end{pmatrix}, \quad \text{such that} \quad \boldsymbol{\Psi}(\boldsymbol{p}) = \mathcal{B}(\boldsymbol{\varphi})\boldsymbol{\Psi}(0), \tag{7.2.5}$$

where $\mathcal{B}(\boldsymbol{\varphi})$ is the boost operator of the $(1,0) \oplus (0,1)$ representation:

$$\mathcal{B}(\boldsymbol{\varphi}) \equiv \begin{pmatrix} \exp\left[+\boldsymbol{J}\cdot\boldsymbol{\varphi}\right] & \mathbb{O}_3 \\ \mathbb{O}_3 & \exp\left[-\boldsymbol{J}\cdot\boldsymbol{\varphi}\right] \end{pmatrix}. \tag{7.2.6}$$

With the boost operator thus defined the consistency of our earlier choice of parity matrix can be readily checked. Using the explicit form of β given in (7.1.7), along with (7.2.6), we find

$$\beta\mathcal{B}(\boldsymbol{\varphi})\beta^{-1} = \mathcal{B}(-\boldsymbol{\varphi}). \tag{7.2.7}$$

Hence, the matrix β inverts the direction of the rapidity vector; that is, it inverts the unit three-momentum as expected. Applying β to $\boldsymbol{\Psi}(\boldsymbol{p})$ we obtain

$$\beta\boldsymbol{\Psi}(\boldsymbol{p}) = \beta\mathcal{B}(\boldsymbol{\varphi})\beta^{-1}\beta\boldsymbol{\Psi}(0) = \mathcal{B}(-\boldsymbol{\varphi})\boldsymbol{\Psi}(0) = \boldsymbol{\Psi}(-\boldsymbol{p}). \tag{7.2.8}$$

We here encounter the first consequence of the choice $\boldsymbol{X}(0) = \boldsymbol{Y}(0)$ in (7.2.4). If we had chosen instead $\boldsymbol{X}(0) = -\boldsymbol{Y}(0)$, then (7.2.8) would read $\beta\boldsymbol{\Psi}(\boldsymbol{p}) = -\boldsymbol{\Psi}(-\boldsymbol{p})$. If the field $\boldsymbol{\Psi}(\boldsymbol{p})$ were to be given the interpretation of a particle, akin to Ryder's interpretation of the Weyl spinors [82, p. 41], then one might say that $\beta\boldsymbol{\Psi}(\boldsymbol{p}) = +\boldsymbol{\Psi}(-\boldsymbol{p})$ describes a particle of positive intrinsic parity whereas $\beta\boldsymbol{\Psi}(\boldsymbol{p}) = -\boldsymbol{\Psi}(-\boldsymbol{p})$ describes a particle of negative intrinsic parity. Such an interpretation does not seem viable, however, because the field $\boldsymbol{\Psi}(\boldsymbol{p})$ does not transform unitarily under the $(1,0) \oplus (0,1)$ representation of $\mathcal{L}^{\uparrow}_{+}$. This is a direct consequence of the non-compactness of the restricted Lorentz group [82, p. 40].

Now, in order to obtain the field equation for $\boldsymbol{\Psi}(\boldsymbol{p})$, it will be convenient to first establish some identities for the constituent fields $\boldsymbol{X}(\boldsymbol{p})$ and $\boldsymbol{Y}(\boldsymbol{p})$. Using (7.2.1) and (7.2.2) along with (7.2.4) we can relate $\boldsymbol{X}(\boldsymbol{p})$ and $\boldsymbol{Y}(\boldsymbol{p})$ as follows:

$$\boldsymbol{X}(\boldsymbol{p}) = \exp[+2\boldsymbol{J}\cdot\boldsymbol{\varphi}]\boldsymbol{Y}(\boldsymbol{p}), \tag{7.2.9}$$

$$\boldsymbol{Y}(\boldsymbol{p}) = \exp[-2\boldsymbol{J}\cdot\boldsymbol{\varphi}]\boldsymbol{X}(\boldsymbol{p}). \tag{7.2.10}$$

Performing a Maclaurin series expansion of $\exp[2\,\boldsymbol{J}\cdot\boldsymbol{\varphi}]$ the resulting expression can be written in closed form using the identity $(\boldsymbol{J}\cdot\hat{\boldsymbol{p}})^3 = (\boldsymbol{J}\cdot\hat{\boldsymbol{p}})$. Explicitly, we have

$$\begin{aligned}
&\exp[2\boldsymbol{J}\cdot\boldsymbol{\varphi}]\\
&= \mathbb{1}_3 + (2\boldsymbol{J}\cdot\boldsymbol{\varphi})^1 + \frac{1}{2}(2\boldsymbol{J}\cdot\boldsymbol{\varphi})^2 + \frac{1}{3!}(2\boldsymbol{J}\cdot\boldsymbol{\varphi})^3 + \frac{1}{4!}(2\boldsymbol{J}\cdot\boldsymbol{\varphi})^4 + \dots\\
&= \mathbb{1}_3 + (\boldsymbol{J}\cdot\hat{\boldsymbol{p}})(2\varphi) + \frac{1}{2}(\boldsymbol{J}\cdot\hat{\boldsymbol{p}})^2(2\varphi)^2 + \frac{1}{3!}(\boldsymbol{J}\cdot\hat{\boldsymbol{p}})^3(2\varphi)^3 + \frac{1}{4!}(\boldsymbol{J}\cdot\hat{\boldsymbol{p}})^4(2\varphi)^4 + \dots\\
&= \mathbb{1}_3 + (\boldsymbol{J}\cdot\hat{\boldsymbol{p}})\left[(2\varphi) + \frac{1}{3!}(2\varphi)^3 + \dots\right] + (\boldsymbol{J}\cdot\hat{\boldsymbol{p}})^2\left[\frac{1}{2}(2\varphi)^2 + \frac{1}{4!}(2\varphi)^4 + \dots\right]\\
&= \mathbb{1}_3 + (\boldsymbol{J}\cdot\hat{\boldsymbol{p}})\sinh(2\varphi) + (\boldsymbol{J}\cdot\hat{\boldsymbol{p}})^2\left[\cosh(2\varphi) - 1\right]. \qquad (7.2.11)
\end{aligned}$$

Recalling the double angle formulae

$$\sinh(2\varphi) = 2\sinh(\varphi)\cosh(\varphi) \quad \text{and} \quad \cosh(2\varphi) = 2\sinh^2(\varphi) + 1, \qquad (7.2.12)$$

and inserting (7.2.3) we obtain

$$\exp[2\boldsymbol{J}\cdot\boldsymbol{\varphi}] = \mathbb{1}_3 + 2(\boldsymbol{J}\cdot\hat{\boldsymbol{p}})\frac{|\boldsymbol{p}|}{m}\frac{E}{m} + 2(\boldsymbol{J}\cdot\hat{\boldsymbol{p}})^2\frac{|\boldsymbol{p}|^2}{m^2}. \qquad (7.2.13)$$

Finally, we absorb $|\boldsymbol{p}|$ and isolate a factor of inverse mass squared such that

$$\exp[2\boldsymbol{J}\cdot\boldsymbol{\varphi}] = \frac{1}{m^2}\left[m^2\mathbb{1}_3 + 2(\boldsymbol{J}\cdot\boldsymbol{p})E + 2(\boldsymbol{J}\cdot\boldsymbol{p})^2\right]. \qquad (7.2.14)$$

With this identity, (7.2.9) and (7.2.10) become

$$[m^2\mathbb{1}_3 + 2(\boldsymbol{J}\cdot\boldsymbol{p})^2 + 2(\boldsymbol{J}\cdot\boldsymbol{p})E]\boldsymbol{Y}(\boldsymbol{p}) = m^2\boldsymbol{X}(\boldsymbol{p}), \qquad (7.2.15)$$

$$[m^2\mathbb{1}_3 + 2(\boldsymbol{J}\cdot\boldsymbol{p})^2 - 2(\boldsymbol{J}\cdot\boldsymbol{p})E]\boldsymbol{X}(\boldsymbol{p}) = m^2\boldsymbol{Y}(\boldsymbol{p}). \qquad (7.2.16)$$

We may now express these relations in a manifestly Lorentz covariant from as an operator equation on $\boldsymbol{\Psi}(\boldsymbol{p})$. Rewriting (7.2.15) and (7.2.16) such that all the terms on the LHS of the equals sign are in terms of the components of p^μ we obtain

$$[p^\mu p_\mu\mathbb{1}_3 + 2(\boldsymbol{J}\cdot\boldsymbol{p})^2 + 2(\boldsymbol{J}\cdot\boldsymbol{p})p_0]\boldsymbol{Y}(\boldsymbol{p}) = m^2\boldsymbol{X}(\boldsymbol{p}), \qquad (7.2.17)$$

$$[p^\mu p_\mu\mathbb{1}_3 + 2(\boldsymbol{J}\cdot\boldsymbol{p})^2 - 2(\boldsymbol{J}\cdot\boldsymbol{p})p_0]\boldsymbol{X}(\boldsymbol{p}) = m^2\boldsymbol{Y}(\boldsymbol{p}). \qquad (7.2.18)$$

Looking at [119, p. 98] or [64, App. B] we find the identity

$$\gamma_{\mu\nu}p^\mu p^\nu = \begin{pmatrix} \mathbb{O}_3 & p^\mu p_\mu\mathbb{1}_3 + 2(\boldsymbol{J}\cdot\boldsymbol{p})^2 + 2(\boldsymbol{J}\cdot\boldsymbol{p})p_0 \\ p^\mu p_\mu\mathbb{1}_3 + 2(\boldsymbol{J}\cdot\boldsymbol{p})^2 - 2(\boldsymbol{J}\cdot\boldsymbol{p})p_0 & \mathbb{O}_3 \end{pmatrix}.$$

where $\gamma_{\mu\nu}$ are the gamma matrices of the $(1,0)\oplus(0,1)$ representation:

$$\begin{aligned}\gamma^{00} &= \beta,\\ \gamma^{0i} &= \gamma^{i0} = \mathfrak{J}_i\gamma_0\beta,\\ \gamma^{ij} &= \{\mathfrak{J}_i,\mathfrak{J}_j\}\,\beta - \delta^i{}_j\beta,\end{aligned}$$

with $(\delta^i{}_j) = \mathbb{1}_3$ and

$$\mathfrak{J}_i \equiv \begin{pmatrix}\mathcal{J}_i & \mathbb{O}_3\\ \mathbb{O}_3 & \mathcal{J}_i\end{pmatrix} \quad \text{and} \quad \gamma_5 \equiv \begin{pmatrix}\mathbb{1}_3 & \mathbb{O}_3\\ \mathbb{O}_3 & -\mathbb{1}_3\end{pmatrix}. \tag{7.2.19}$$

These gamma matrices satisfy the algebra [64]:

$$\left\{\gamma^{\mu\rho},\gamma^{\nu\lambda}\right\} + \left\{\gamma^{\mu\nu},\gamma^{\rho\lambda}\right\} + \left\{\gamma^{\mu\lambda},\gamma^{\rho\nu}\right\} = 2\left[g^{\mu\nu}g^{\rho\lambda} + g^{\mu\rho}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\rho}\right], \tag{7.2.20}$$

where $g^{\mu\nu}$ is the Minkowski metric. Using the above identity, (7.2.17) and (7.2.18) now take the manifestly Lorentz covariant form

$$\left(\gamma_{\mu\nu}p^\mu p^\nu - m^2\mathbb{1}_6\right)\boldsymbol{\Psi}(\boldsymbol{p}) = 0. \tag{7.2.21}$$

We immediately recognise this as the relativistic spin one wave equation in momentum space [60, 64]. At this point we again remark on the choice $\boldsymbol{X}(0) = \boldsymbol{Y}(0)$. Had we instead taken $\boldsymbol{X}(0) = -\boldsymbol{Y}(0)$, the matrix operator in (7.2.21) would appear with a different sign in front of the mass term: $\left(\gamma_{\mu\nu}p^\mu p^\nu + m^2\mathbb{1}_6\right)$. We will return to this point shortly once the Proca equations have been identified.

Before we proceed to the next section let us briefly explore the earlier mentioned "kinematic acausality" of [119, 120, 122–126]. It is a mathematical requirement for the existence of non-trivial solutions that the determinant of the operator to the left of the field $\boldsymbol{\Psi}(\boldsymbol{p})$ must vanish. A summary of the analysis given, for instance, in [121] is provided in Tab. 7.1.

	Matrix operator	Determinant	Solutions
(a)	$\gamma_{\mu\nu}p^\mu p^\nu - m^2\mathbb{1}_6$	$[m^4 - (E^2 - \lvert\boldsymbol{p}\rvert^2)^2]^3$	$E^2 = \lvert\boldsymbol{p}\rvert^2 \pm m^2$
(b)	$\gamma_{\mu\nu}p^\mu p^\nu$	$-[E^2 - \lvert\boldsymbol{p}\rvert^2]^6$	$E^2 = \lvert\boldsymbol{p}\rvert^2$
(c)	$(\boldsymbol{J}\cdot\boldsymbol{p})^2 \pm (\boldsymbol{J}\cdot\boldsymbol{p})\,p_0$	0	
(d)	$(\boldsymbol{J}\cdot\boldsymbol{p}) \pm p_0\mathbb{1}_3$	$\pm E\left(E^2 - \lvert\boldsymbol{p}\rvert^2\right)$	$E = 0, \quad E^2 = \lvert\boldsymbol{p}\rvert^2$

Table 7.1: Matrix operators: (a) is the matrix operator derived above in (7.2.21); (b) is obtained from (7.2.21) by setting $m = 0$; (c) is obtained from (b) by setting $p^\mu p_\mu = 0$ in keeping with the choice $m = 0$; (d) is the operator reported in [119, p. 105] to yield Maxwell's equations.

The reported "kinematic acausality" of [119, 120, 122–126] is based upon the observation that not all solutions of the determinants in Tab. 7.1 are $E^2 = |\boldsymbol{p}|^2 + m^2$, in the massive case, or $E^2 = |\boldsymbol{p}|^2$, in the massless case. In particular, (a) admits the acausal solution $E^2 = |\boldsymbol{p}|^2 - m^2$; (b) admits only causal solutions; (c) admits acausal solutions because no constraint is placed on the relation between energy and momentum; (d) admits the acausal solution $E = 0$ for any value of momentum. Consequently, the operator in (b) is put forward [121, p. 9] as the correct operator to be employed in the derivation the equations of electromagnetism in free-space; although, under the assumed [119] direct identification of electric and magnetic fields with the linear combinations of the $(1, 0)$ and $(0, 1)$ fields, the equations obtained from the operator in (b) do not coincide with the free-space Maxwell [122].

We here provide a different interpretation. As is clear from the above derivation of the relativistic spin one field equation, the causal relation $E^2 = |\boldsymbol{p}|^2 + m^2$ is imposed at the very outset, namely in the parametrisation of the rapidity given in (7.2.3). It is therefore inconsistent to later consider other relations between energy, three-momentum, and mass. One might just as well have completely avoided the shorthand notation E and used instead $\sqrt{|\boldsymbol{p}|^2 + m^2}$ throughout the derivation of (7.2.21). In that case the determinant calculated above in (a) would vanish identically. The same is true of the other operators in Tab. 7.1. In other words, the above analysis shows that no further constraint on the relation between energy, momentum, and mass, beyond what has already been assumed, is needed in order to ensure that the here considered mathematical prerequisite for the existence of non-trivial solutions is met. A result to the contrary would simply mean that further constraints are needed or that there are no non-trivial solutions of the matrix operator in question.

7.3 Even and odd parity linear combinations

In order to show that there exists a consistent identification of the components of $\boldsymbol{\Psi}(p)$ whereby the above derived $\left(\gamma_{\mu\nu} p^\mu p^\nu - m^2 \mathbb{1}_6\right) \boldsymbol{\Psi}(\boldsymbol{p}) = 0$ corresponds to the Proca equation and, in the massless limit, to Maxwell's equations, we must first converting to coordinate space. Using the Fourier transformation

$$\Phi(x^\mu) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d^4p \, e^{-ip\cdot x} \Phi(p^\mu) \qquad \Rightarrow \qquad p_\mu \to i\,\partial_\mu,$$

the field equation becomes

$$\left(\gamma_{\mu\nu} \partial^\mu \partial^\nu + m^2 \mathbb{1}_6\right) \boldsymbol{\Psi}(x) = 0. \tag{7.3.1}$$

Likewise (7.2.17) and (7.2.18) become

$$\left[-\partial^\mu \partial_\mu \mathbb{1}_3 - 2(\boldsymbol{J}\cdot\boldsymbol{\nabla})^2 + 2(\boldsymbol{J}\cdot\boldsymbol{\nabla})\partial_t\right] \boldsymbol{Y}(x) = m^2 \boldsymbol{X}(x), \tag{7.3.2}$$

$$\left[-\partial^\mu \partial_\mu \mathbb{1}_3 - 2(\boldsymbol{J}\cdot\boldsymbol{\nabla})^2 - 2(\boldsymbol{J}\cdot\boldsymbol{\nabla})\partial_t\right] \boldsymbol{X}(x) = m^2 \boldsymbol{Y}(x). \tag{7.3.3}$$

Considering that the electric field transforms as a vector whereas the magnetic field transforms as pseudovector under space-inversion it will prove favourable to write (7.3.2) and (7.3.3) in terms of the even and odd linear combinations of $\boldsymbol{X}$ and $\boldsymbol{Y}$. We thus add and subtract (7.3.2) and (7.3.3) to obtain

$$-2(\boldsymbol{J}\cdot\boldsymbol{\nabla})^2\left[\boldsymbol{X}+\boldsymbol{Y}\right]-2(\boldsymbol{J}\cdot\boldsymbol{\nabla})\partial_t\left[\boldsymbol{X}-\boldsymbol{Y}\right]=\left(\partial^\mu\partial_\mu+m^2\right)\left[\boldsymbol{X}+\boldsymbol{Y}\right], \quad (7.3.4)$$

$$-2(\boldsymbol{J}\cdot\boldsymbol{\nabla})^2\left[\boldsymbol{X}-\boldsymbol{Y}\right]-2(\boldsymbol{J}\cdot\boldsymbol{\nabla})\partial_t\left[\boldsymbol{X}+\boldsymbol{Y}\right]=\left(\partial^\mu\partial_\mu-m^2\right)\left[\boldsymbol{X}-\boldsymbol{Y}\right]. \quad (7.3.5)$$

Here, the explicit spacetime index has been omitted and the Klein-Gordon term has been isolated on the right. It follows from the dispersion relation that $\boldsymbol{X}$ and $\boldsymbol{Y}$ must satisfy the Klein-Gordon equation; hence, (7.3.4) and (7.3.5) become

$$-2(\boldsymbol{J}\cdot\boldsymbol{\nabla})^2\left[\boldsymbol{X}+\boldsymbol{Y}\right]-2(\boldsymbol{J}\cdot\boldsymbol{\nabla})\partial_t\left[\boldsymbol{X}-\boldsymbol{Y}\right]=0, \quad (7.3.6)$$

$$-2(\boldsymbol{J}\cdot\boldsymbol{\nabla})^2\left[\boldsymbol{X}-\boldsymbol{Y}\right]-2(\boldsymbol{J}\cdot\boldsymbol{\nabla})\partial_t\left[\boldsymbol{X}+\boldsymbol{Y}\right]=-2m^2\left[\boldsymbol{X}-\boldsymbol{Y}\right]. \quad (7.3.7)$$

Isolating like terms we obtain

$$(\boldsymbol{J}\cdot\boldsymbol{\nabla})\left[(\boldsymbol{J}\cdot\boldsymbol{\nabla})\left[\boldsymbol{X}+\boldsymbol{Y}\right]+\partial_t\left[\boldsymbol{X}-\boldsymbol{Y}\right]\right]=0, \quad (7.3.8)$$

$$(\boldsymbol{J}\cdot\boldsymbol{\nabla})\left[(\boldsymbol{J}\cdot\boldsymbol{\nabla})\left[\boldsymbol{X}-\boldsymbol{Y}\right]+\partial_t\left[\boldsymbol{X}+\boldsymbol{Y}\right]\right]=m^2\left[\boldsymbol{X}-\boldsymbol{Y}\right]. \quad (7.3.9)$$

Thanks to our earlier choice of the adjoint representation we may now avail ourselves of the identity

$$\begin{aligned}\boldsymbol{J}\cdot\boldsymbol{\nabla}\equiv&\mathcal{J}_x\partial_x+\mathcal{J}_y\partial_y+\mathcal{J}_z\partial_z\\ =&i\begin{pmatrix}0&-\partial_z&\partial_y\\ \partial_z&0&-\partial_x\\ -\partial_y&\partial_x&0\end{pmatrix}=i\begin{pmatrix}0&\epsilon_{xyz}\partial_z&\epsilon_{xzy}\partial_y\\ \epsilon_{yxz}\partial_z&0&\epsilon_{yzx}\partial_x\\ \epsilon_{zxy}\partial_y&\epsilon_{zyx}\partial_x&0\end{pmatrix}=i\left(\epsilon_{ijk}\partial_k\right).\end{aligned}$$

Expressing (7.3.8) and (7.3.9) in component form

$$(\boldsymbol{J}\cdot\boldsymbol{\nabla})_{ij}\left[(\boldsymbol{J}\cdot\boldsymbol{\nabla})_{jl}\left[\boldsymbol{X}+\boldsymbol{Y}\right]_l+\partial_t\left[\boldsymbol{X}-\boldsymbol{Y}\right]_j\right]=0,$$

$$(\boldsymbol{J}\cdot\boldsymbol{\nabla})_{ij}\left[(\boldsymbol{J}\cdot\boldsymbol{\nabla})_{jl}\left[\boldsymbol{X}-\boldsymbol{Y}\right]_l+\partial_t\left[\boldsymbol{X}+\boldsymbol{Y}\right]_j\right]=m^2\left[\boldsymbol{X}-\boldsymbol{Y}\right]_i,$$

and inserting the above identity we obtain

$$i\epsilon_{ijk}\partial_k\left[i\epsilon_{jlm}\partial_m\left[\boldsymbol{X}+\boldsymbol{Y}\right]_l+\partial_t\left[\boldsymbol{X}-\boldsymbol{Y}\right]_j\right]=0, \quad (7.3.10)$$

$$i\epsilon_{ijk}\partial_k\left[i\epsilon_{jlm}\partial_m\left[\boldsymbol{X}-\boldsymbol{Y}\right]_l+\partial_t\left[\boldsymbol{X}+\boldsymbol{Y}\right]_j\right]=m^2\left[\boldsymbol{X}-\boldsymbol{Y}\right]_i. \quad (7.3.11)$$

Using $\boldsymbol{\nabla} \times \boldsymbol{X} = -(\epsilon_{ijk}\partial_j X_k) = (\epsilon_{ijk}\partial_k X_j)$, to return to vector notation, (7.3.10) and (7.3.11) become

$$i\boldsymbol{\nabla} \times [i\boldsymbol{\nabla} \times [\boldsymbol{X}+\boldsymbol{Y}] + \partial_t [\boldsymbol{X}-\boldsymbol{Y}]] = 0, \tag{7.3.12}$$

$$i\boldsymbol{\nabla} \times [i\boldsymbol{\nabla} \times [\boldsymbol{X}-\boldsymbol{Y}] + \partial_t [\boldsymbol{X}+\boldsymbol{Y}]] = m^2 [\boldsymbol{X}-\boldsymbol{Y}]. \tag{7.3.13}$$

Multiplying by appropriate factors and rearranging slightly one finally obtains

$$\boldsymbol{\nabla} \times [i\boldsymbol{\nabla} \times [\boldsymbol{X}+\boldsymbol{Y}]] + \partial_t [\boldsymbol{\nabla} \times [\boldsymbol{X}-\boldsymbol{Y}]] = 0, \tag{7.3.14}$$

$$\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times [\boldsymbol{X}-\boldsymbol{Y}]] - \partial_t [i\boldsymbol{\nabla} \times [\boldsymbol{X}+\boldsymbol{Y}]] = -m^2 [\boldsymbol{X}-\boldsymbol{Y}]. \tag{7.3.15}$$

These are the dynamical equations for the even and odd parity linear combinations of the three-component fields $\boldsymbol{X}$ and $\boldsymbol{Y}$. To clarify the identification of the Proca equations it will be useful to briefly review their covariant formulation in terms of the electromagnetic field strength tensor $F^{\mu\nu}$.

7.4 Review of the covariant formulation

The covariant formulation of the Proca equations in terms of the completely antisymmetric field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ reads [82, p. 68]

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \quad \text{and} \quad \left(\partial_\mu \partial^\mu + m^2\right) A^\nu = 0. \tag{7.4.1}$$

In the case of $m \neq 0$ these equations manifestly fail to be gauge-invariant because the choice $\partial_\mu A^\mu = 0$, the Lorentz gauge, is forced upon us by the antisymmetry of $F^{\mu\nu}$ [82, p. 67].

In order to recover the Proca equations in vector notation we choose the components of the field strength tensor to be $F^{i0} = -E^i$ and $F^{ij} = -\epsilon^{ijk} B_k$. Furthermore, choosing $\boldsymbol{\mathcal{E}} \equiv (E_i)$, $\boldsymbol{\mathcal{B}} \equiv (B_i)$, and $(A^\mu) = (A^0, A^i) \equiv (A^0, \boldsymbol{A})$ the equations read

$$\begin{aligned} m^2 A^0 &= -\partial_\mu F^{\mu 0} = -\partial_i F^{i0} \\ &= +\partial_i E^i \qquad\qquad \Longleftrightarrow \qquad\qquad m^2 A^0 = -\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}}, \end{aligned} \tag{7.4.2}$$

$$\begin{aligned} m^2 A^i &= -\,\partial_\mu F^{\mu i} = -\partial_0 F^{0i} - \partial_j F^{ji} \\ &= -\partial_t\, E^i - \epsilon^{ijk}\partial_j B_k \qquad \Longleftrightarrow \qquad m^2 \boldsymbol{A} = +\partial_t\, \boldsymbol{\mathcal{E}} - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}}. \end{aligned} \tag{7.4.3}$$

The remaining four equations follow from the Bianchi identity

$$\partial_\alpha F_{\sigma\kappa} + \partial_\sigma F_{\kappa\alpha} + \partial_\kappa F_{\alpha\sigma} = 0.$$

This identity follows trivially from the definition of the field strength tensor in terms of the vector potential under the assumption that partial derivatives commute on A^μ. The constraint equation for the divergence of $\boldsymbol{\mathcal{B}}$ is obtained from the Bianchi identity by taking all three indices to be spatial. Choosing $\alpha = i$, $\sigma = j$, and $\kappa = l$ we obtain

$$\partial_i F_{jl} + \partial_j F_{li} + \partial_l F_{ij} = 0. \tag{7.4.4}$$

Substituting $F^{ij} = -\epsilon^{ijk} B_k$ this becomes

$$-\epsilon_{jlk}\partial_i B^k - \epsilon_{lik}\partial_j B^k - \epsilon_{ijk}\partial_l B^k = 0.$$

Contracting with ϵ^{jlm} and using the identity $\epsilon^{ijk}\epsilon_{imn} = -\left(\delta^j{}_m\delta^k{}_n - \delta^j{}_n\delta^k{}_m\right)$, see App. A, we find

$$\begin{aligned}
0 &= -\epsilon^{jlm}\epsilon_{jlk}\partial_i B^k - \epsilon^{jlm}\epsilon_{lik}\partial_j B^k - \epsilon^{jlm}\epsilon_{ijk}\partial_l B^k \\
&= -\epsilon^{jlm}\epsilon_{jlk}\partial_i B^k + \epsilon^{ljm}\epsilon_{lik}\partial_j B^k + \epsilon^{jlm}\epsilon_{jik}\partial_l B^k \\
&= (\delta^l{}_l\delta^m{}_k - \delta^l{}_k\delta^m{}_l)\partial_i B^k - (\delta^j{}_i\delta^m{}_k - \delta^j{}_k\delta^m{}_i)\partial_j B^k - (\delta^l{}_i\delta^m{}_k - \delta^l{}_k\delta^m{}_i)\partial_l B^k \\
&= 3\,\partial_i B^m - \partial_i B^m - \partial_i B^m + \delta^m{}_i\partial_k B^k - \partial_i B^m + \delta^m{}_i\partial_k B^k \\
&= 2\,\delta^m{}_i\partial_k B^k \qquad \iff \qquad 0 = \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{B}}.
\end{aligned} \tag{7.4.5}$$

The three remaining equations are obtained by choosing one of the indices in the Bianchi identity to be temporal. This is the only remaining non-trivial choice of indices. Taking any two indices to be the same gives an expression that vanishes identically on account of the antisymmetry of $F_{\mu\nu}$. Choosing $\alpha = 0$, $\sigma = j$, and $\kappa = k$ the Bianchi identity becomes

$$\partial_0 F_{jk} + \partial_j F_{k0} + \partial_k F_{0j} = 0. \tag{7.4.6}$$

Substituting $F^{i0} = -E^i$ and $F^{ij} = -\epsilon^{ijk}B_k$, contracting with ϵ_{jkm}, and using the identity $\epsilon^{jkm}\epsilon_{jkn} = -2\,\delta^m{}_n$, see App. A, we obtain

$$\begin{aligned}
0 &= -\epsilon_{jkl}\partial_0 B^l - \partial_j E_k + \partial_k E_j \\
&= -\epsilon^{jkm}\epsilon_{jkl}\partial_0 B^l - \epsilon^{jkm}\partial_j E_k + \epsilon^{jkm}\partial_k E_j \\
&= 2\,\delta^m{}_l\partial_0 B^l - \epsilon^{jkm}\left(\partial_j E_k - \partial_k E_j\right) \\
&= 2\,\partial_0 B^m - 2\,\epsilon^{mjk}\partial_j E_k \qquad \iff \qquad 0 = \partial_t\boldsymbol{\mathcal{B}} + \boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}}.
\end{aligned} \tag{7.4.7}$$

The Proca equations in vector form have thus been recovered from the covariant form given above in (7.4.1).

Two further useful expressions are the solutions of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$, respectively, in terms of the vector potential. From the definition of the field strength tensor in terms of the derivatives of the vector potential we find

$$E_i = F_{0i} = \partial_0 A_i - \partial_i A_0 \qquad \iff \qquad \boldsymbol{\mathcal{E}} = -\partial_t \boldsymbol{A} - \boldsymbol{\nabla} A_0. \tag{7.4.8}$$

Similarly, taking F_{ij}, we have

$$-\epsilon_{ijk}B^k = F_{ij} = \partial_i A_j - \partial_j A_i.$$

Contracting with ϵ^{ijm} we thus obtain

$$2B^m = 2\epsilon^{mij}\partial_i A_j \qquad \Longleftrightarrow \qquad \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{7.4.9}$$

These relations will now be employed toward the identification of the Proca equations from (7.3.14) and (7.3.15).

7.5 Identification of the Proca and Maxwell equations

The dynamical relations derived in Sec. 7.3 read

$$\boldsymbol{\nabla} \times [i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}]] + \partial_t [\boldsymbol{\nabla} \times [\boldsymbol{X} - \boldsymbol{Y}]] = 0, \tag{7.5.1}$$

$$\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times [\boldsymbol{X} - \boldsymbol{Y}]] - \partial_t [i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}]] = -m^2 [\boldsymbol{X} - \boldsymbol{Y}]. \tag{7.5.2}$$

Making the identification $(\boldsymbol{X} - \boldsymbol{Y}) = \boldsymbol{A}$ and recalling, from (7.4.9), that $\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \boldsymbol{A}$ we obtain

$$\boldsymbol{\nabla} \times [i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}]] + \partial_t \boldsymbol{\mathcal{B}} = 0, \tag{7.5.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \partial_t [i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}]] = -m^2 \boldsymbol{A}. \tag{7.5.4}$$

The only remaining choice we have is for $i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}]$. Taking this to be equal to $\boldsymbol{\mathcal{E}}$ yields

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} + \partial_t \boldsymbol{\mathcal{B}} = 0, \tag{7.5.5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \partial_t \boldsymbol{\mathcal{E}} = -m^2 \boldsymbol{A}. \tag{7.5.6}$$

We recognise these as the six dynamical Proca equations as recited in (7.4.7) and (7.4.3), respectively. By (7.4.6), (7.4.7), and (7.4.3) these are related to the covariant formulation via

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} + \partial_t \boldsymbol{\mathcal{B}} = 0 \qquad \Longleftrightarrow \qquad \partial_0 F_{jk} + \partial_j F_{k0} + \partial_k F_{0j} = 0, \tag{7.5.7}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \partial_t \boldsymbol{\mathcal{E}} = -m^2 \boldsymbol{A} \qquad \Longleftrightarrow \qquad \partial_\mu F^{\mu i} = -m^2 A^i. \tag{7.5.8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \cdot [\boldsymbol{\nabla} \times [\boldsymbol{X} - \boldsymbol{Y}]] = 0, \tag{7.5.9}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} = \boldsymbol{\nabla} \cdot [i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}]] = 0. \tag{7.5.10}$$

By (7.4.4), (7.4.5), and (7.4.2) these are related to the covariant formulation via

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}} = 0 \qquad \Longleftrightarrow \qquad \partial_i F_{jl} + \partial_j F_{li} + \partial_l F_{ij} = 0, \tag{7.5.11}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} = 0 \qquad \Longleftrightarrow \qquad \partial_\mu F^{\mu 0} = -m^2 A^0, \tag{7.5.12}$$

in the Weyl gauge $A^0 = 0$. It should not be surprising that the equations are obtained in a particular gauge because, as noted in the preceding section, the Proca equations are not gauge-invariant; consequently, the equations here derived must come in one gauge or another.

The Klein-Gordon equation for A^μ, with $A^0 = 0$, follows from identification $\boldsymbol{A} = \boldsymbol{X} - \boldsymbol{Y}$ because $\boldsymbol{X}$ and $\boldsymbol{Y}$ satisfy this equation.

Taking the massless limit of (7.5.7), (7.5.8), (7.5.11), and (7.5.12) we obtain the free-space Maxwell equations:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} + \partial_t \boldsymbol{\mathcal{B}} = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}} = 0 \qquad &\Longleftrightarrow \qquad 0 = \partial_\alpha F_{\sigma\kappa} + \partial_\sigma F_{\kappa\alpha} + \partial_\kappa F_{\alpha\sigma}, \\
\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \partial_t \boldsymbol{\mathcal{E}} = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} = 0 \qquad &\Longleftrightarrow \qquad 0 = \partial_\mu F^{\mu\nu}.
\end{aligned}$$

We have thus established that under the identification of the linear combinations of the fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ given by

$$\boldsymbol{\mathcal{E}} = i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}], \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times [\boldsymbol{X} - \boldsymbol{Y}], \quad \text{and} \quad \boldsymbol{A} = \boldsymbol{X} - \boldsymbol{Y}, \tag{7.5.13}$$

the relativistic spin one field equation $\left(\gamma_{\mu\nu}\partial^\mu\partial^\nu + m^2 \mathbb{1}_6\right) \boldsymbol{\Psi}(x) = 0$, as expanded in (7.3.14) and (7.3.15), yields the Proca equations in the Weyl gauge. These become the Maxwell equations in the limit $m \to 0$.

Recalling that $\boldsymbol{X}$ and $\boldsymbol{Y}$ were chosen such that they would be exchanged under space-inversion, it follows that $\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}]$ and $\boldsymbol{X} - \boldsymbol{Y}$ both transform as a vectors whereas $\boldsymbol{\nabla} \times [\boldsymbol{X} - \boldsymbol{Y}]$ transforms as a pseudovector under space-inversion. This is consistent with the known properties of $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{B}}$, and $\boldsymbol{A}$ under space-inversion. If we had chosen not $\boldsymbol{X}(0) = \boldsymbol{Y}(0)$ but $\boldsymbol{X}(0) = -\boldsymbol{Y}(0)$, then (7.3.2) and (7.3.3) would both differ by a minus sign. In turn (7.3.4) and (7.3.5) would differ by a minus sign and the derivation would manifestly fail at (7.5.2) where we would find that there exists no consistent identification of $\boldsymbol{A}$ in terms of linear combinations of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Before we conclude let us briefly show how (7.3.14) and (7.3.15) can be related directly to the covariant Proca equations given in (7.4.1). Recalling the relation between the electric and magnetic fields and the vector potential, as given in (7.4.8) and (7.4.9),

$$\boldsymbol{\mathcal{E}} = -\partial_t \boldsymbol{A} - \boldsymbol{\nabla} A^0 \quad \text{and} \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \boldsymbol{A},$$

we may express $\boldsymbol{\nabla} A^0$ and $\boldsymbol{A}$ in terms of $\boldsymbol{X}$ and $\boldsymbol{Y}$:

$$\boldsymbol{\nabla} A^0 = -i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}] - \partial_t [\boldsymbol{X} - \boldsymbol{Y}], \tag{7.5.14}$$

$$\boldsymbol{A} = \boldsymbol{X} - \boldsymbol{Y}. \tag{7.5.15}$$

Hence, from (7.3.15) and solutions (7.5.14) and (7.5.15), we have

$$\begin{aligned}
0 &= m^2 [\boldsymbol{X} - \boldsymbol{Y}] - i\boldsymbol{\nabla} \times [i\boldsymbol{\nabla} \times [\boldsymbol{X} - \boldsymbol{Y}] + \partial_t [\boldsymbol{X} + \boldsymbol{Y}]] \\
&= m^2 \boldsymbol{A} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} + \partial_t \left(\partial_t \boldsymbol{A} + \boldsymbol{\nabla} A^0\right) \\
&= m^2 \boldsymbol{A} + \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} + \partial_t \left(\partial_t \boldsymbol{A} + \boldsymbol{\nabla} A^0\right) \\
&= m^2 \boldsymbol{A} + \partial_t^2 \boldsymbol{A} - \nabla^2 \boldsymbol{A} + \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) + \boldsymbol{\nabla} \left(\partial_t A^0\right) \\
&= m^2 \boldsymbol{A} + \partial_\mu \partial^\mu \boldsymbol{A} + \boldsymbol{\nabla} \left(\partial_\mu A^\mu\right) \\
&= m^2 \boldsymbol{A} + \partial_\mu \left(\partial^\mu \boldsymbol{A} + \boldsymbol{\nabla} A^\mu\right) \\
&= m^2 A^i + \partial_\mu \left(\partial^\mu A^i - \partial^i A^\mu\right) \\
&= m^2 A^i + \partial_\mu F^{\mu i}.
\end{aligned} \tag{7.5.16}$$

This is (7.4.3). The other equation we require is for the divergence of $\boldsymbol{\mathcal{E}}$ as per (7.4.2). Noting that (7.5.14) is a solution of (7.3.14), and that $\boldsymbol{\mathcal{E}}$ is given in (7.5.14) by $i\boldsymbol{\nabla} \times [\boldsymbol{X} + \boldsymbol{Y}]$, we take the divergence of (7.5.14) to obtain

$$\begin{aligned}
0 &= -\nabla^2 A^0 - \partial_t \boldsymbol{\nabla} \cdot \boldsymbol{A} \\
&= \partial_t \partial_t A^0 - \nabla^2 A^0 - \partial_t \partial_t A^0 - \partial_t \boldsymbol{\nabla} \cdot \boldsymbol{A} \\
&= \left(\partial_0 \partial^0 + \partial_i \partial^i\right) A^0 - \partial_t \left(\partial_0 A^0 + \partial_i A^i\right) \\
&= \partial_\mu \partial^\mu A^0 - \partial_t \left(\partial_\mu A^\mu\right) \\
&= \partial_\mu \left(\partial^\mu A^0 - \partial^0 A^\mu\right) \\
&= \partial_\mu F^{\mu 0}.
\end{aligned} \tag{7.5.17}$$

Thus, in the Weyl gauge $A^0 = 0$, we have the Proca equation $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$; the equation $\left(\partial_\mu \partial^\mu + m^2\right) A^\nu = 0$ follows trivially because $\boldsymbol{X}$ and $\boldsymbol{Y}$ satisfy this equation. Using (7.3.15) and recalling that the divergence of a curl is zero we also have $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. This is no surprise considering $A^0 = 0$ and the earlier mentioned consequence of the structure of the Proca equations, namely, that $\partial_\mu A^\mu = 0$.

In the case $m = 0$ we may have a non-zero value of A^0. The gauge-invariant Maxwell equations thus follow from (7.5.16) and (7.5.17):

$$\partial_\mu F^{\mu\nu} = 0. \tag{7.5.18}$$

Given that the free-space Maxwell equations can be derived, via the here proposed identification, from the massive relativistic spin one field equation in the limit $m \to 0$ we conclude that there is no discrepancy with the result obtained by Weinberg [66] in the case $m = 0$.

7.6 Conclusion

We have demonstrated that there exists a consistent identification of the curls of the even and odd linear combinations of the components, $\boldsymbol{X}$ and $\boldsymbol{Y}$, of the six-component classical field $\boldsymbol{\Psi}(x)$ whereby the relativistic spin one field equation $\left(\partial_\mu\partial^\mu + m^2\right)\boldsymbol{\Psi}(x) = 0$ yields the Proca equations $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$ and $\left(\partial_\mu\partial^\mu + m^2\right)A^\nu = 0$ in the Weyl gauge. In the massless limit these equations reduce to the free-space Maxwell equations $\partial_\mu F^{\mu\nu} = 0$. It follows that there is no discrepancy between the relativistic spin one field equations obtained by Weinberg in the case $m = 0$ and the massless limit of the massive spin one field equations. The here provided identification of the curls of the even and odd linear combinations of the fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ was found to be consistent with the known transformation properties of electric and magnetic fields under space-inversion. Furthermore, it was argued that the here derived field equations are free of "kinematic acausality." This is true in the massive case and in the massless case.

A

Notation and conventions

Indices

Greek letters: $\mu, \nu, \ldots$ range over values $0, 1, 2, \ldots$.

Latin letters: $i, j, k, \ldots$ range over values $1, 2, 3, \ldots$.

Units

The speed of light and the reduced Planck constant are taken to be: $c = \hbar = 1$.

Metric

The Minkowski metric $\eta^{\mu\nu}$ is diagonal and has non-zero elements $\eta^{00} = -\eta^{ii} = 1$. The components of $\eta_{\mu\nu}$ are identical to those of $\eta^{\mu\nu}$. These matrices serve as raising and lowering operators, respectively; hence,

$$\eta^{\mu}{}_{\nu} = \eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^{\mu}{}_{\nu}.$$

Unless otherwise stated, we will use the Einstein summation convention whereby any repeated index, one up, one down, is summed.

Kronecker delta

In 3-dimensions: $\delta^{i}{}_{j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$

In 4-dimensions: $\delta^{\mu}{}_{\nu} = \begin{cases} 1 & \mu = \nu, \\ 0 & \mu \neq \nu. \end{cases}$

4-Vectors

Covariant vectors: $x^{\mu} = (x^0, x^i) \equiv (t, \boldsymbol{x})$.

Contravariant vectors: $x_{\mu} = (x_0, x_i) \equiv (t, -\boldsymbol{x})$.

Fourier transform

$$\Psi(x^\mu) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} d^4p\, e^{-ip\cdot x}\Psi(p^\mu), \quad \Rightarrow \quad p_\mu \to i\,\partial_\mu,$$

$$\Psi(p^\mu) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} d^4x\, e^{+ip\cdot x}\Psi(x^\mu).$$

The normalisation in D dimensions is $1/(2\pi)^{D/2}$.

Step function

$$\theta(t) = \begin{cases} 1 & t > 0, \\ 0 & t < 0. \end{cases} \tag{A.0.1}$$

Fourier representation of step function

$$\theta(t) = \frac{-1}{2\pi i}\int_{-\infty}^{+\infty} d\omega \frac{\mathrm{e}^{-i\omega t}}{\omega + i\epsilon}. \tag{A.0.2}$$

Levi-Civita Symbol
The completely antisymmetric Levi-Civita symbol is defined as follows:

$$\text{In 3-dimensions: } \epsilon^{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 1\ 2\ 3, \\ -1 & \text{if } ijk \text{ is an odd permutation of } 1\ 2\ 3, \\ 0 & \text{if any two indices are repeated.} \end{cases}$$

$$\text{In 4-dimensions: } \epsilon^{\alpha\beta\sigma\rho} = \begin{cases} +1 & \text{if } \alpha\beta\sigma\rho \text{ is an even permutation of } 0\ 1\ 2\ 3, \\ -1 & \text{if } \alpha\beta\sigma\rho \text{ is an odd permutation of } 0\ 1\ 2\ 3, \\ 0 & \text{if any two indices are repeated.} \end{cases}$$

Some useful identities

$$\epsilon^{\alpha\beta\sigma\rho}\,\epsilon_{\mu\nu\kappa\lambda} = -\det\begin{pmatrix} \delta^{\alpha}{}_{\mu} & \delta^{\alpha}{}_{\nu} & \delta^{\alpha}{}_{\kappa} & \delta^{\alpha}{}_{\lambda} \\ \delta^{\beta}{}_{\mu} & \delta^{\beta}{}_{\nu} & \delta^{\beta}{}_{\kappa} & \delta^{\beta}{}_{\lambda} \\ \delta^{\sigma}{}_{\mu} & \delta^{\sigma}{}_{\nu} & \delta^{\sigma}{}_{\kappa} & \delta^{\sigma}{}_{\lambda} \\ \delta^{\rho}{}_{\mu} & \delta^{\rho}{}_{\nu} & \delta^{\rho}{}_{\kappa} & \delta^{\rho}{}_{\lambda} \end{pmatrix}, \tag{A.0.3}$$

$$\epsilon^{\alpha\beta\sigma\rho} = -\epsilon_{\alpha\beta\sigma\rho}, \tag{A.0.4}$$

$$\epsilon^{0ijk} = \epsilon^{ijk}, \tag{A.0.5}$$

$$\epsilon^{ijk} = -\epsilon_{ijk}, \tag{A.0.6}$$

$$\epsilon^{ijk}\epsilon_{lmn} = -\det\begin{pmatrix} \delta^{i}{}_{l} & \delta^{i}{}_{m} & \delta^{i}{}_{n} \\ \delta^{j}{}_{l} & \delta^{j}{}_{m} & \delta^{j}{}_{n} \\ \delta^{k}{}_{l} & \delta^{k}{}_{m} & \delta^{k}{}_{n} \end{pmatrix} \tag{A.0.7}$$

$$= -\delta^{i}{}_{l}\left(\delta^{j}{}_{m}\delta^{k}{}_{n} - \delta^{j}{}_{n}\delta^{k}{}_{m}\right) \tag{A.0.8}$$

$$+\,\delta^{i}{}_{m}\left(\delta^{j}{}_{l}\delta^{k}{}_{n} - \delta^{j}{}_{n}\delta^{k}{}_{l}\right) \tag{A.0.9}$$

$$-\,\delta^{i}{}_{n}\left(\delta^{j}{}_{l}\delta^{k}{}_{m} - \delta^{j}{}_{m}\delta^{k}{}_{l}\right), \tag{A.0.10}$$

$$\epsilon^{ijk}\epsilon_{imn} = -\left(\delta^{j}{}_{m}\delta^{k}{}_{n} - \delta^{j}{}_{n}\delta^{k}{}_{m}\right), \tag{A.0.11}$$

$$\epsilon^{ijk}\epsilon_{ijn} = -\,2\,\delta^{k}{}_{n}. \tag{A.0.12}$$

B

Explicit expressions, identities, and derivations

We here attend to the derivation of various explicit expressions and identities that should not be altogether omitted for the sake of completeness. They are here relegated to an appendix on the grounds that they would otherwise disrupt the smooth flow of the text.

B.1 The Wigner rotation

The Wigner rotation is defined as a particular succession of Lorentz transformations that leaves the standard vector, k, invariant. It is given by

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p)\,\Lambda\,L(p). \tag{B.1.1}$$

Here $L(p)$ is defined as the boost that takes the standard vector k to four momentum p. The defining relation thus reads

$$p = L(p)k. \tag{B.1.2}$$

Of course $L(p)$ implicitly depends on the choice of k. We are here interested in the case of a massive particle of positive energy where, as per Tab. 2.1, we have $(k^\mu) = (m, 0, 0, 0)$. The four-momentum is given by $(p^\mu) = \left(p^0, \boldsymbol{p}\right)$ where $p^0 = \sqrt{|\boldsymbol{p}|^2 + m^2}$. The explicit matrix components of $L(p)$ are derived in App. B.2.1. This defines the first and the last term in (B.1.1). The remaining term, Λ, is an arbitrary Lorentz transformation that takes p to Λp.

Weinberg [40, p. 59] shows that in the special case where Λ is a pure rotation, $\mathscr{R}$, the Wigner rotation is given by

$$W(\mathscr{R}, p) \equiv L^{-1}(\mathscr{R}p)\,\mathscr{R}\,L(p) = \mathscr{R}. \tag{B.1.3}$$

We here show that in the case where Λ is a arbitrary boost, $\mathscr{B}$, we again obtain a rotation, namely the rotation that is required in composing two boosts into a single boost and a rotation.

Let $\Lambda = \mathscr{B}$. The Wigner rotation then reads

$$W(\mathscr{B}, p) \equiv L^{-1}(\mathscr{B}p)\,\mathscr{B}\,L(p). \tag{B.1.4}$$

Considering that any two boosts can be expressed as a single boost and a rotation, we may rewrite the first two terms in (B.1.4) as

$$\mathscr{R}L(q) = \mathscr{B}L(p), \tag{B.1.5}$$

where $\mathscr{R}$ and q are implicitly defined by this relation. In fact, we can do better than this and write the parameter q explicitly in terms of p and the here defined operations. Applying (B.1.5) to the standard vector, we have

$$\mathscr{R}L(q)k = \mathscr{B}L(p)k. \tag{B.1.6}$$

But $\mathscr{R}L(q)k = \mathscr{R}q$ and $\mathscr{B}L(p)k = \mathscr{B}p$; therefore, $\mathscr{R}q = \mathscr{B}p$ and we obtain

$$q = \mathscr{R}^{-1}\mathscr{B}p. \tag{B.1.7}$$

Substitution into (B.1.5) yields the identity

$$\mathscr{R}L(\mathscr{R}^{-1}\mathscr{B}p) = \mathscr{B}L(p). \tag{B.1.8}$$

Inserting (B.1.8) into (B.1.4) gives

$$\begin{aligned} W(\mathscr{B}, p) &= L^{-1}(\mathscr{B}p)\,\mathscr{B}\,L(p) \\ &= L^{-1}(\mathscr{B}L(p)k)\,\mathscr{B}\,L(p) \\ &= L^{-1}(\mathscr{R}L(\mathscr{R}^{-1}\mathscr{B}p)k)\,\mathscr{R}L(\mathscr{R}^{-1}\mathscr{B}p). \end{aligned} \tag{B.1.9}$$

Using (B.1.7) this becomes

$$\begin{aligned} W(\mathscr{B}, p) &= L^{-1}(\mathscr{R}L(q)k)\,\mathscr{R}L(q) \\ &= L^{-1}(\mathscr{R}q)\,\mathscr{R}L(q). \end{aligned} \tag{B.1.10}$$

As pointed out by Weinberg [40, p. 68] that the standard boost $L(q)$ may be expressed as

$$L(q) = R(\hat{\boldsymbol{q}})B(|\boldsymbol{q}|)R^{-1}(\hat{\boldsymbol{q}}), \tag{B.1.11}$$

where $R^{-1}(\hat{\boldsymbol{q}})$ rotates from $\hat{\boldsymbol{q}}$ into $\hat{\boldsymbol{z}}$, $B(|\hat{\boldsymbol{q}}|)$ boosts along $\hat{\boldsymbol{z}}$ to magnitude $|\hat{\boldsymbol{q}}|$, and $R(\hat{\boldsymbol{q}})$ rotates back into the direction $\hat{\boldsymbol{q}}$. Accordingly, the inverse standard boost, $L^{-1}(\mathscr{R}q)$, is simply

$$L^{-1}(\mathscr{R}q) = R(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}})B^{-1}(|\hat{\boldsymbol{q}}|)R^{-1}(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}}). \tag{B.1.12}$$

Substituting (B.1.11) and (B.1.12) into (B.1.10) gives

$$W(\mathscr{B}, p) = R(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}})B^{-1}(|\hat{\boldsymbol{q}}|)R^{-1}(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}})\mathscr{R}R(\hat{\boldsymbol{q}})B(|\hat{\boldsymbol{q}}|)R^{-1}(\hat{\boldsymbol{q}}). \tag{B.1.13}$$

The term $R^{-1}(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}})\mathscr{R}R(\hat{\boldsymbol{q}})$, in (B.1.13), is a succession of rotations that takes $\hat{\boldsymbol{z}}$ into the direction $\hat{\boldsymbol{q}}$, then into the direction $\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}}$, and back into $\hat{\boldsymbol{z}}$; hence, it is a rotation about the z-axis [40, p. 69]. This term, therefore, commutes with $B(|\hat{\boldsymbol{q}}|)$ by the definition of $B(|\hat{\boldsymbol{q}}|)$

as a boost along the z-axis. Making use of this observation, (B.1.13) becomes

$$\begin{aligned} W(\mathscr{B},p) &= R(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}})B^{-1}(|\hat{\boldsymbol{q}}|)B(|\hat{\boldsymbol{q}}|)R^{-1}(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}})\mathscr{R}R(\hat{\boldsymbol{q}})R^{-1}(\hat{\boldsymbol{q}}) \\ &= R(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}})R^{-1}(\boldsymbol{\mathscr{R}}\hat{\boldsymbol{q}})\mathscr{R} \\ &= \mathscr{R}. \end{aligned} \tag{B.1.14}$$

Hence, from the definition of $\mathscr{R}$ as given in (B.1.8), we have shown that $W(\mathscr{B},p)$ is the rotation that is required in the composition of two successive boosts into a single boost and a rotation. It follows that $W(\mathscr{B},p)$ is equal to the identity transformation if and only if the two boosts $L(p)$ and $\mathscr{B}$ are colinear.

B.2 The standard boost operator

The standard boost operator $L(p)$ is defined above, in (B.1.2), as a Lorentz transformation[1] that maps a given standard vector k^μ to momentum p^μ. We here derive the matrix elements of $L(p)$ for the two physical scenarios considered in Sec. 2.1.1.

B.2.1 Massive particle of positive energy

In the case of a massive particle of positive energy the standard vector, as given in Tab. 2.1, reads $(k^\mu) = \big(m,0,0,0\big)$ where m is the mass of the particle concerned. The momentum vector we wish to obtained from the standard vector via $L(p)$ is, of course, $(p^\mu) = (\sqrt{\boldsymbol{p}^2+m^2},\boldsymbol{p})$. From this it follows, as will now be verified, that the standard boost operator is given by

$$L(p) = \exp[i\boldsymbol{K}\cdot\boldsymbol{\varphi}], \tag{B.2.1}$$

where $\boldsymbol{K} \equiv (K_x,K_y,K_z)$, a set three infinitesimal generators of Lorentz boost. The rapidity vector $\boldsymbol{\varphi} \equiv \varphi\,\hat{\boldsymbol{\varphi}}$ is aligned with the direction of the three-momentum $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{p}}$ and its magnitude φ is parameterised by

$$\cosh(\varphi) = E/m = \gamma \quad \text{and} \quad \sinh(\varphi) = |\boldsymbol{p}|/m = \gamma\beta, \tag{B.2.2}$$

where $E = \sqrt{\boldsymbol{p}^2+m^2}$; $\gamma = \big(1-\beta^2\big)^{-1/2}$ with $\beta = |\boldsymbol{x}|/c$, the ratio of the magnitude of the velocity and the speed of light. The infinitesimal generators $\boldsymbol{K}$ are those of the vector

[1] For given k^μ and p^μ the Lorentz transformation $L(p)$ is unique only up to redefinition through right multiplication by an element of the little group.

basis. These read

$$K_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_y = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_z = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{B.2.3}$$

The Maclaurin series expansion of (B.2.1), by the result (C.2.3) and the observation $(i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^3 = i\boldsymbol{K}\cdot\hat{\boldsymbol{p}}$, reads

$$\exp[i\boldsymbol{K}\cdot\boldsymbol{\varphi}] = \mathbb{1}_4 + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\,[\sinh(\varphi)] + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\,[\cosh(\varphi) - 1]\,. \tag{B.2.4}$$

Here, by explicit calculation using (B.2.3), we have

$$i\boldsymbol{K}\cdot\hat{\boldsymbol{p}} = \begin{pmatrix} 0 & \hat{p}_x & \hat{p}_y & \hat{p}_z \\ \hat{p}_x & 0 & 0 & 0 \\ \hat{p}_y & 0 & 0 & 0 \\ \hat{p}_z & 0 & 0 & 0 \end{pmatrix}, \quad (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \hat{p}_x\hat{p}_x & \hat{p}_x\hat{p}_y & \hat{p}_x\hat{p}_z \\ 0 & \hat{p}_y\hat{p}_x & \hat{p}_y\hat{p}_y & \hat{p}_y\hat{p}_z \\ 0 & \hat{p}_z\hat{p}_x & \hat{p}_z\hat{p}_y & \hat{p}_z\hat{p}_z \end{pmatrix}, \tag{B.2.5}$$

where $\hat{p}_x \equiv p^1/\,|\boldsymbol{p}|$, $\hat{p}_y \equiv p^2/\,|\boldsymbol{p}|$, and $\hat{p}_z \equiv p^3/\,|\boldsymbol{p}|$. Using (B.2.5) in (B.2.4) we may thus express the standard boost operator $L(p)$ succinctly in terms of its components $L(p)^\mu{}_\nu$ as

$$L(p)^0{}_0 = \cosh(\varphi), \tag{B.2.6}$$

$$L(p)^i{}_0 = L(p)^0{}_i = \hat{p}^i \sinh(\varphi), \tag{B.2.7}$$

$$L(p)^i{}_j = \delta^i{}_j + \hat{p}^i\hat{p}^j\,[\cosh(\varphi) - 1]\,, \tag{B.2.8}$$

where $\cosh(\varphi)$ and $\sinh(\varphi)$ are given in (B.2.2).

We may now verify that $L(p)^\mu{}_\nu$ maps k^μ to p^μ. Inserting $\cosh(\varphi) = \sqrt{\boldsymbol{p}^2 + m^2}/m$ and $\sinh(\varphi) = |\boldsymbol{p}|/m$ into (B.2.6)–(B.2.8), these become

$$L(p)^0{}_0 = \sqrt{\boldsymbol{p}^2 + m^2}/m, \tag{B.2.9}$$

$$L(p)^i{}_0 = L(p)^0{}_i = p^i/m, \tag{B.2.10}$$

$$L(p)^i{}_j = \delta^i{}_j + \hat{p}^i\hat{p}^j\left[\sqrt{\boldsymbol{p}^2 + m^2}/m - 1\right]. \tag{B.2.11}$$

Computing $L(p)^{\mu}{}_{\nu}k^{\nu}$, we thus obtain

$$L(p)^{0}{}_{\nu}k^{\nu} = \sqrt{\boldsymbol{p}^2 + m^2},$$

$$L(p)^{i}{}_{\nu}k^{\nu} = p^{i}.$$

Hence $L(p)$, as derived above, maps $(k^{\mu}) = (m, 0, 0, 0)$ to $(p^{\mu}) = (\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p})$ as was to be shown.

B.2.2 Massless particle of positive energy

The standard vector for a massless particle of positive energy, as given in Tab. 2.1, reads $(k^{\mu}) = (\kappa, 0, 0, \kappa)$. We here derive an explicit expression for the Lorentz transformation $L(p)$ that maps the standard vector to $(p^{\mu}) = (|\boldsymbol{p}|, \boldsymbol{p})$, as per (B.1.2). The crucial difference between the former case of a massive particle and the present one is, of course, that a massless particle cannot be brought to rest by any Lorentz transformation. Hence, for the three-vector component of the standard vector, we have $\boldsymbol{k}^T = (0, 0, \kappa)$ where κ is a positive non-zero real number, the momentum of the particle. There is thus an orientation associated with k^{μ}, a circumstance that must be taken into account in the formulation of $L(p)$. In the present case, where the standard vector has been chosen to be aligned with the z-axis, it will be most convenient to define $L(p)$ in terms of a boost $B(|\boldsymbol{p}|)$ along the z-axis from κ to momentum $|\boldsymbol{p}|$ and a subsequent rotation $R(\hat{\boldsymbol{p}})$ from the z-axis into the direction $\hat{\boldsymbol{p}}$ [65]. The standard boost is thus decomposed as

$$L(p) = R(\hat{\boldsymbol{p}})B(|\boldsymbol{p}|). \tag{B.2.12}$$

We shall begin by deriving $B(|\boldsymbol{p}|)$. This is given by

$$B(|\boldsymbol{p}|) = \exp[iK_z\varphi], \quad \text{with} \quad \varphi \equiv \log\left(\frac{|\boldsymbol{p}|}{\kappa}\right), \tag{B.2.13}$$

where K_z is the generator of Lorentz boost along the z-axis as given above in (B.2.3). Exploiting the expansion given above in (B.2.4) by setting $\hat{\boldsymbol{p}}^T = (0, 0, 1)$, we have

$$\exp[iK_z\varphi] = \mathbb{1}_4 + (iK_z)\left[\sinh(\varphi)\right] + (iK_z)^2\left[\cosh(\varphi) - 1\right]. \tag{B.2.14}$$

The components of the boost $B(|\boldsymbol{p}|)$ thus read

$$B(|\boldsymbol{p}|)^{0}{}_{0} = B(|\boldsymbol{p}|)^{3}{}_{3} = \cosh(\varphi), \tag{B.2.15}$$

$$B(|\boldsymbol{p}|)^{1}{}_{1} = B(|\boldsymbol{p}|)^{2}{}_{2} = 1, \tag{B.2.16}$$

$$B(|\boldsymbol{p}|)^{0}{}_{3} = B(|\boldsymbol{p}|)^{3}{}_{0} = \sinh(\varphi). \tag{B.2.17}$$

Inserting the definition of the rapidity parameter, this becomes

$$B(|\boldsymbol{p}|)^0{}_0 = B(|\boldsymbol{p}|)^3{}_3 = \left(|\boldsymbol{p}|^2 + \kappa^2\right)/2\kappa|\boldsymbol{p}|, \tag{B.2.18}$$

$$B(|\boldsymbol{p}|)^1{}_1 = B(|\boldsymbol{p}|)^2{}_2 = 1, \tag{B.2.19}$$

$$B(|\boldsymbol{p}|)^0{}_3 = B(|\boldsymbol{p}|)^3{}_0 = \left(|\boldsymbol{p}|^2 - \kappa^2\right)/2\kappa|\boldsymbol{p}|. \tag{B.2.20}$$

We may now compute the components of $B(|\boldsymbol{p}|)^\mu{}_\nu k^\nu$ as

$$\begin{aligned}
B(|\boldsymbol{p}|)^0{}_\nu k^\nu &= B(|\boldsymbol{p}|)^0{}_0 k^0 + B(|\boldsymbol{p}|)^0{}_3 k^3 \\
&= \frac{|\boldsymbol{p}|^2 + \kappa^2}{2\kappa|\boldsymbol{p}|}\kappa + \frac{|\boldsymbol{p}|^2 - \kappa^2}{2\kappa|\boldsymbol{p}|}\kappa = |\boldsymbol{p}|, \\
B(|\boldsymbol{p}|)^1{}_\nu k^\nu &= B(|\boldsymbol{p}|)^1{}_0 k^0 + B(|\boldsymbol{p}|)^1{}_3 k^3 = 0, \\
B(|\boldsymbol{p}|)^2{}_\nu k^\nu &= B(|\boldsymbol{p}|)^2{}_0 k^0 + B(|\boldsymbol{p}|)^2{}_3 k^3 = 0, \\
B(|\boldsymbol{p}|)^3{}_\nu k^\nu &= B(|\boldsymbol{p}|)^3{}_0 k^0 + B(|\boldsymbol{p}|)^3{}_3 k^3 \\
&= \frac{|\boldsymbol{p}|^2 - \kappa^2}{2\kappa|\boldsymbol{p}|}\kappa + \frac{|\boldsymbol{p}|^2 + \kappa^2}{2\kappa|\boldsymbol{p}|}\kappa = |\boldsymbol{p}|.
\end{aligned} \tag{B.2.21}$$

As promised, $B(|\boldsymbol{p}|)$ maps $(k^\mu) = (\kappa, 0, 0, \kappa)$ to $(p^\mu) = (|\boldsymbol{p}|, 0, 0, |\boldsymbol{p}|)$.

We shall now derive $R(\hat{\boldsymbol{p}})$, but first we require the Minkowski space rotation operators. These are obtained via the exponential map from the infinitesimal generators

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.2.22}$$

We thus have

$$R(\theta_k) = \exp[iJ_k\theta_k], \qquad k \in \{x, y, z\}, \tag{B.2.23}$$

where θ_k is the angle of rotation about the k-axis. Computing the cube of the infinitesimal generators (B.2.22) we find $(iJ_k)^3 = -iJ_k$, allowing us to use (C.1.3) in order to express the series expansion of (B.2.23) in closed from as

$$R(\theta_k) = \mathbb{1}_4 + (iJ_k)\left[\sin(\theta_k)\right] + (iJ_k)^2\left[1 - \cos(\theta_k)\right]. \tag{B.2.24}$$

In component form, the three rotation operators thus read, for a rotation about the x-axis

$$R(\theta_x)^0{}_0 = R(\theta_x)^1{}_1 = 1, \tag{B.2.25}$$

$$R(\theta_x)^2{}_2 = R(\theta_x)^3{}_3 = \cos(\theta_x), \tag{B.2.26}$$

$$R(\theta_x)^3{}_2 = -R(\theta_x)^2{}_3 = -\sin(\theta_x), \tag{B.2.27}$$

for a rotation about the y-axis

$$R(\theta_y)^0{}_0 = R(\theta_y)^2{}_2 = 1, \tag{B.2.28}$$

$$R(\theta_y)^1{}_1 = R(\theta_y)^3{}_3 = \cos(\theta_x), \tag{B.2.29}$$

$$R(\theta_y)^3{}_1 = -R(\theta_y)^1{}_3 = \sin(\theta_x), \tag{B.2.30}$$

and for a rotation about the z-axis

$$R(\theta_z)^0{}_0 = R(\theta_z)^3{}_3 = 1, \tag{B.2.31}$$

$$R(\theta_z)^1{}_1 = R(\theta_z)^2{}_2 = \cos(\theta_x), \tag{B.2.32}$$

$$R(\theta_z)^2{}_1 = -R(\theta_z)^1{}_2 = -\sin(\theta_x). \tag{B.2.33}$$

There are many ways one might go about defining $R(\hat{\boldsymbol{p}})$ using some multiple of the above rotation operators depending on the parametrisation of $\hat{\boldsymbol{p}}$ in terms of the angles θ_k. We shall here write $\hat{\boldsymbol{p}}$ such that the angels θ_k may be identified with the polar and azimuthal angles, $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$, of a spherical coordinate system. Taking

$$p_x = |\boldsymbol{p}| \cos(\theta) \sin(\phi), \quad p_y = |\boldsymbol{p}| \sin(\theta) \sin(\phi), \quad \text{and} \quad p_z = |\boldsymbol{p}| \cos(\phi), \tag{B.2.34}$$

we obtain

$$R(\hat{\boldsymbol{p}}) = R(\theta) R(\phi), \tag{B.2.35}$$

where $R(\phi)$ is a rotation about the y-axis by an angle negative ϕ; $R(\theta)$ is a rotation about the z-axis by an angle negative θ. With these identifications we have in component form

$$R(\hat{\boldsymbol{p}})^0{}_0 = 1, \quad R(\hat{\boldsymbol{p}})^1{}_1 = \cos(\theta)\cos(\phi), \quad R(\hat{\boldsymbol{p}})^2{}_2 = \cos(\theta), \quad R(\hat{\boldsymbol{p}})^3{}_3 = \cos(\phi),$$

$$R(\hat{\boldsymbol{p}})^2{}_1 = \cos(\phi)\cos(\theta), \quad R(\hat{\boldsymbol{p}})^1{}_2 = -\sin(\theta),$$

$$R(\hat{\boldsymbol{p}})^3{}_1 = -\sin(\theta), \quad R(\hat{\boldsymbol{p}})^1{}_3 = \cos(\theta)\sin(\phi),$$

$$R(\hat{\boldsymbol{p}})^2{}_3 = \sin(\theta)\sin(\phi).$$

To verify that $L(p) = R(\hat{\boldsymbol{p}})B(|\boldsymbol{p}|)$, as derived above, maps k^μ to p^μ we compute $p^\mu = R(\hat{\boldsymbol{p}})^\mu{}_\nu B(|\boldsymbol{p}|)^\nu{}_\alpha k^\alpha$ as follows. Recalling from (B.2.21) that

$$B(|\boldsymbol{p}|)^0{}_\alpha k^\alpha = B(|\boldsymbol{p}|)^3{}_\alpha k^\alpha = |\boldsymbol{p}| \quad \text{and} \quad B(|\boldsymbol{p}|)^1{}_\alpha k^\alpha = B(|\boldsymbol{p}|)^2{}_\alpha k^\alpha = 0,$$

it follows that

$$R(\hat{\boldsymbol{p}})^{\mu}{}_{\nu}B(|\boldsymbol{p}|)^{\nu}{}_{\alpha}k^{\alpha} = R(\hat{\boldsymbol{p}})^{\mu}{}_{0}B(|\boldsymbol{p}|)^{0}{}_{\alpha}k^{\alpha} + R(\hat{\boldsymbol{p}})^{\mu}{}_{3}B(|\boldsymbol{p}|)^{3}{}_{\alpha}k^{\alpha} \tag{B.2.36}$$

$$= |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{\mu}{}_{0} + |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{\mu}{}_{3}. \tag{B.2.37}$$

We thus obtain

$$L(p)^{0}{}_{\mu}k^{\mu} = |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{0}{}_{0} + |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{0}{}_{3} = |\boldsymbol{p}|,$$

$$L(p)^{1}{}_{\mu}k^{\mu} = |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{1}{}_{0} + |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{1}{}_{3} = |\boldsymbol{p}|\cos(\theta)\sin(\phi),$$

$$L(p)^{2}{}_{\mu}k^{\mu} = |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{2}{}_{0} + |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{2}{}_{3} = |\boldsymbol{p}|\sin(\theta)\sin(\phi),$$

$$L(p)^{3}{}_{\mu}k^{\mu} = |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{3}{}_{0} + |\boldsymbol{p}|R(\hat{\boldsymbol{p}})^{3}{}_{3} = |\boldsymbol{p}|\cos(\phi).$$

The Lorentz transformation $L(p) = R(\hat{\boldsymbol{p}})B(|\boldsymbol{p}|)$ thus maps $(k^{\mu}) = (\kappa, 0, 0, \kappa)$ to $(p^{\mu}) = (|\boldsymbol{p}|, \boldsymbol{p})$ where $\boldsymbol{p}$ is expressed in polar coordinates as defined above in (B.2.34). This completes our derivation of the standard boost operator for a massless particle of positive energy.

B.3 The standard boost and discrete symmetries

Using the explicit form of the standard boost for a massive particles of positive energy, given above in (B.2.6)–(B.2.8), and the discrete symmetries on Minkowski space, defined in (2.1.5), we now prove the identities $\mathscr{P}L(p)\mathscr{P}^{-1} = L(\mathscr{P}p)$ and $\mathscr{T}L(p)\mathscr{T}^{-1} = L(\mathscr{P}p)$. In component form we have

$$\left(\mathscr{P}L(p)\mathscr{P}^{-1}\right)_{\mu\nu} = \mathscr{P}_{\mu}{}^{\alpha}\mathscr{P}_{\nu}{}^{\beta}L(p)_{\alpha\beta}$$

Looking separately at each of the components specified in (B.2.6)–(B.2.8) we find

$$\left(\mathscr{P}L(p)\mathscr{P}^{-1}\right)_{00} = \mathscr{P}_{0}{}^{\alpha}\mathscr{P}_{0}{}^{\beta}L(p)_{\alpha\beta} = \mathscr{P}_{0}{}^{0}\mathscr{P}_{0}{}^{0}L(p)_{00} = L(p)_{00},$$

$$\left(\mathscr{P}L(p)\mathscr{P}^{-1}\right)_{i0} = \mathscr{P}_{i}{}^{\alpha}\mathscr{P}_{0}{}^{\beta}L(p)_{\alpha\beta} = \mathscr{P}_{i}{}^{j}\mathscr{P}_{0}{}^{0}L(p)_{j0} = -L(p)_{i0},$$

$$\left(\mathscr{P}L(p)\mathscr{P}^{-1}\right)_{ij} = \mathscr{P}_{i}{}^{\alpha}\mathscr{P}_{j}{}^{\beta}L(p)_{\alpha\beta} = \mathscr{P}_{i}{}^{k}\mathscr{P}_{j}{}^{l}L(p)_{kl} = L(p)_{ij}.$$

Consequently

$$\left(\mathscr{P}L(p)\mathscr{P}^{-1}\right)^{0}{}_{0} = L(p)^{0}{}_{0} = \cosh(\varphi),$$

$$\left(\mathscr{P}L(p)\mathscr{P}^{-1}\right)^{i}{}_{0} = -L(p)^{i}{}_{0} = -L(p)^{0}{}_{i} = -\hat{p}^{i}\sinh(\varphi),$$

$$\left(\mathscr{P}L(p)\mathscr{P}^{-1}\right)^{i}{}_{j} = L(p)^{i}{}_{j} = \delta^{i}{}_{j} + \hat{p}^{i}\hat{p}^{j}\left[\cosh(\varphi) - 1\right].$$

Hence, as required, we obtain

$$\mathscr{P} L(p) \mathscr{P}^{-1} = L(\mathscr{P} p). \tag{B.3.1}$$

Noting from (2.1.5) that $\mathscr{T} = -\mathscr{P}$, we can write $\mathscr{T} L(p) \mathscr{T}^{-1} = \mathscr{P} L(p) \mathscr{P}^{-1}$ and the remaining identity follows trivially from (B.3.1) to read

$$\mathscr{T} L(p) \mathscr{T}^{-1} = L(\mathscr{P} p). \tag{B.3.2}$$

B.4 The propagator

The propagator $-i\Delta_{lm}$ is defined as the pairing of a field $\psi_l(x)$ with its dual $\bar{\psi}_m$ [40, p. 274]. This is equivalent to the vacuum expectation value of the time ordered product of $\psi_l(x)$ and $\bar{\psi}_m$ given by

$$-i\Delta_{lm}(x,y) = \theta(x^0 - y^0)\langle\ \ |\psi_l(x)\bar{\psi}_m(y)|\ \ \rangle \pm \theta(y^0 - x^0)\langle\ \ |\bar{\psi}_m(y)\psi_l(x)|\ \ \rangle, \tag{B.4.1}$$

where the minus sign in '$\pm$' applies in the case of a fermionic field; the step function $\theta(x^0 - y^0)$, defined in App. A, is equal to zero for $x^0 < y^0$, whereas, it is equal to one for $x^0 > y^0$. The dual field $\bar{\psi}_m(x)$ is related to $\psi_m(x)$ by Hermitian conjugation and a conjugate-linear mapping of the expansion coefficients, as per App. B.5. The field, as defined in Sec. 2.5, is given by

$$\psi(x) \equiv \kappa\, \psi^+(x) + \lambda\, \psi^{c-}(x), \quad \text{with} \quad \kappa, \lambda \in \mathbb{C}, \tag{B.4.2}$$

and

$$\psi_l^+(x) = (2\pi)^{-3/2} \int d^3p\, \mathrm{e}^{-ip\cdot x} \sum_{\sigma,s} u_l(p;\sigma,s) a(p;\sigma,s), \tag{B.4.3}$$

$$\psi_l^{c-}(x) = (2\pi)^{-3/2} \int d^3p\, \mathrm{e}^{+ip\cdot x} \sum_{\sigma,s} v_l(p;\sigma,s) b^\dagger(p;\sigma,s). \tag{B.4.4}$$

Inserting (B.4.3) and (B.4.4) into (B.4.1) and using the (anti-) commutation relations (2.4.1) and (2.4.2) of the creation and annihilation operators, we obtain

$$\begin{aligned} -\, i\Delta_{lm}(x,y) = {} & \theta(x^0 - y^0)\, (2\pi)^{-3} \int d^3p\, \mathrm{e}^{-ip\cdot(x-y)}\, |\kappa|^2 \sum_{\sigma,s} u_l(p;\sigma,s)\bar{u}_m(p;\sigma,s) \\ & \pm \theta(y^0 - x^0)\, (2\pi)^{-3} \int d^3p\, \mathrm{e}^{-ip\cdot(y-x)}\, |\lambda|^2 \sum_{\sigma,s} v_l(p;\sigma,s)\bar{v}_m(p;\sigma,s). \end{aligned} \tag{B.4.5}$$

The step functions in (B.4.5) have a Fourier representation given in App. A to read

$$\theta(t) = \frac{-1}{2\pi i} \int_{-\infty}^{+\infty} d\omega \frac{\mathrm{e}^{-i\omega t}}{\omega + i\epsilon} \tag{B.4.6}$$

where ω is a free parameter. With this we can rewrite (B.4.5) as the four-dimensional Fourier integral

$$
\begin{aligned}
-\,i\Delta_{lm}(x,y) = i\,(2\pi)^{-3}\int d^4\rho\,\mathrm{e}^{-i\rho\cdot(x-y)}\,\frac{|\kappa|^2 N_{lm}(\boldsymbol{p})}{\rho^0 - p^0 + i\epsilon} \\
\pm\, i\,(2\pi)^{-3}\int d^4\rho\,\mathrm{e}^{-i\rho\cdot(y-x)}\,\frac{|\lambda|^2 M_{lm}(\boldsymbol{p})}{\rho^0 - p^0 + i\epsilon}.
\end{aligned}
\tag{B.4.7}
$$

where $\rho^0 \equiv p^0 + \omega$, $\boldsymbol{\rho} \equiv \boldsymbol{p}$, and the spin sums are succinctly represented by

$$
N_{lm}(\boldsymbol{p}) \equiv \sum_{\sigma,s} u_l(p;\sigma,s)\bar{u}_m(p;\sigma,s), \tag{B.4.8}
$$

$$
M_{lm}(\boldsymbol{p}) \equiv \sum_{\sigma,s} v_l(p;\sigma,s)\bar{v}_m(p;\sigma,s). \tag{B.4.9}
$$

The RHS of (B.4.7) may be further simplified by the change of variables $\rho \to -\rho$ in the second term. This yields

$$
-i\Delta_{lm}(x,y) = i\,(2\pi)^{-3}\int d^4\rho\,\mathrm{e}^{-i\rho\cdot(x-y)}\left[\frac{|\kappa|^2 N_{lm}(\boldsymbol{p})}{\rho^0 - p^0 + i\epsilon} \pm \frac{|\lambda|^2 M_{lm}(-\boldsymbol{p})}{-\rho^0 - p^0 + i\epsilon}\right]. \tag{B.4.10}
$$

Lastly we put the term in the square brackets over a common denominator, isolate a minus sign, and divide by $-i$ on both sides to obtain

$$
\Delta_{lm}(x,y) = (2\pi)^{-3}\int d^4\rho\,\mathrm{e}^{-i\rho\cdot(x-y)}\left[\frac{|\kappa|^2 N_{lm}(\boldsymbol{p})\left(p^0+\rho^0\right) \pm |\lambda|^2 M_{lm}(-\boldsymbol{p})\left(p^0-\rho^0\right)}{-\rho^2 + m^2 - i\epsilon}\right], \tag{B.4.11}
$$

where $\rho^2 \equiv \rho^\mu\rho_\mu = \left(\rho^0\right)^2 - |\boldsymbol{\rho}|^2$. Looking at the RHS of (B.4.11), it is apparent that the functional dependence is purely on the difference $x - y$. We will thus from hence forth denote the propagator by $\Delta(x-y)$ rather than $\Delta(x,y)$.

B.4.1 The Feynman propagator

A special case of the general expression given in (B.4.11) that is of interest in Chs. 3 and 4 is that in which the spin sums $N(\boldsymbol{p})$ and $M(\boldsymbol{p})$ are such that $\Delta(x-y)$ is equal to the Feynman propagator up to an a multiplicative identity matrix. The Feynman propagator is defined in [40, p. 276]. We have

$$
-i\Delta_F(x) \equiv \theta(x^0)\Delta_+(x) + \theta(-x^0)\Delta_+(-x), \tag{B.4.12}
$$

where $\theta(x^0)$ is a step function defined to have value $+1$ for a positive non-zero argument and value zero otherwise; $\Delta_+(x)$ is the standard function given by

$$
\Delta_+(x) = (2\pi)^{-3}\int \frac{d^3p}{2p^0}\,\mathrm{e}^{-ip\cdot x}\,. \tag{B.4.13}
$$

To show how the Feynman propagator is related to (B.4.11), we use the Fourier representation of the step function given above in (B.4.6) to expand the RHS of (B.4.12) as

$$\begin{aligned}
&\theta(x^0)\Delta_+(x) + \theta(-x^0)\Delta_+(-x) \\
&\qquad = i\,(2\pi)^{-4}\int d\omega d^3p \frac{1}{2p^0}\left[\mathrm{e}^{-i\omega x^0}\,\mathrm{e}^{-ip\cdot x}\,\frac{1}{\omega+i\epsilon} + \mathrm{e}^{i\omega x^0}\,\mathrm{e}^{ip\cdot x}\,\frac{1}{\omega+i\epsilon}\right].
\end{aligned}$$

Recalling the parameter ρ^μ with components $\rho^0 = \omega + p^0$ and $\boldsymbol{\rho} = \boldsymbol{p}$, as defined in the text following (B.4.7), we obtain

$$-i\Delta_F(x) = i\,(2\pi)^{-4}\int d^4\rho \frac{1}{2p^0}\left[\mathrm{e}^{-i\rho\cdot x}\,\frac{1}{\rho^0 - p^0 + i\epsilon} + \mathrm{e}^{i\rho\cdot x}\,\frac{1}{\rho^0 - p^0 + i\epsilon}\right].$$

Finally we perform a change of variables to isolate the exponential and subsequently combine the term within the square brackets over a common denominator to obtain

$$\begin{aligned}
-i\Delta_F(x) &= i\,(2\pi)^{-4}\int d^4\rho\,\mathrm{e}^{-i\rho\cdot x}\,\frac{1}{2p^0}\left[\frac{1}{\rho^0 - p^0 + i\epsilon} + \frac{1}{-\rho^0 - p^0 + i\epsilon}\right] \\
&= -i\,(2\pi)^{-4}\int d^4\rho\,\mathrm{e}^{-i\rho\cdot x}\left[\frac{1}{-\rho^2 + m^2 - i\epsilon}\right],
\end{aligned}$$

where $\rho^2 \equiv \rho^\mu\rho_\mu = \left(\rho^0\right)^2 - |\boldsymbol{\rho}|^2$, as in (B.4.11). The Feynman propagator (B.4.12) thus takes the form

$$\Delta_F(x) = (2\pi)^{-4}\int d^4\rho\,\mathrm{e}^{-i\rho\cdot x}\left[\frac{1}{-\rho^2 + m^2 - i\epsilon}\right]. \tag{B.4.14}$$

B.5 The dual space

The present section is devoted to the development of the dual vector space, to be defined below. In the context of the present work, the elements of the dual vector space are obtained via an appropriately defined mapping from the coefficient functions of the field operator. They will be referred to as the dual coefficient functions. The dual field operator is then defined as a field operator expanded in terms of the dual coefficient functions. In order to make these notions precise, we shall recall the relevant mathematical definitions, here taken from Roman [127], Lang [128], [129], and Bogolubov [29].

B.5.1 Definitions and rudimentary development

Definition 1. *Let $\mathcal{F}$ be a field, here chosen to be $\mathbb{R}$ or $\mathbb{C}$; its elements are referred to as scalars. A vector space over $\mathcal{F}$ is a non-empty set $\mathcal{V}$, the elements of which are referred to as vectors, together with two operations: addition of vectors and scalar multiplication. That is, for each pair $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{V}\times\mathcal{V}$ there is assigned a vector $\boldsymbol{x}+\boldsymbol{y} \in \mathcal{V}$ and for each pair $(r,\boldsymbol{x}) \in \mathcal{F}\times\mathcal{V}$ there is assigned a vector $r\boldsymbol{x} \in \mathcal{V}$. Furthermore, the following properties [127, p. 27] must be satisfied for all $r, s \in \mathcal{F}$ and $\boldsymbol{x},\boldsymbol{y} \in \mathcal{V}$.*

1. *Associativity of addition:* $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$.

2. *Commutativity of addition:* $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$.

3. *Existence of a zero:* $\exists\, \mathbf{0} \in \mathcal{V}$ *s.t.* $\mathbf{0} + \boldsymbol{x} = \boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$.

4. *Existence of an additive inverse:* $\exists\, (-\boldsymbol{x}) \in \mathcal{V}$ *s.t.* $\boldsymbol{x} + (-\boldsymbol{x}) = (-\boldsymbol{x}) + \boldsymbol{x} = \mathbf{0}$.

5. *Properties of scalar multiplication:*

$$\begin{aligned} r\,(\boldsymbol{x} + \boldsymbol{y}) &= r\boldsymbol{x} + r\boldsymbol{y}, \\ (r + s)\,\boldsymbol{x} &= r\boldsymbol{x} + s\boldsymbol{x}, \\ (rs)\,\boldsymbol{x} &= r\,(s\boldsymbol{x})\,, \\ 1\boldsymbol{x} &= \boldsymbol{x}. \end{aligned}$$

All vector spaces here considered will be assumed to be finite-dimensional.

Definition 2. *A subspace of a vector space $\mathcal{V}$ is a subset $\mathcal{S}$ of $\mathcal{V}$ that is a vector space in its own right under the operations obtained by restricting the operations of $\mathcal{V}$ to $\mathcal{S}$ [127, p. 29].*

Definition 3. *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. A mapping $g : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is called a sesquilinear from [29, p. 8] if it satisfies the following two properties for all $r, s \in \mathbb{C}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$.*

1. *Conjugate linearity in first coordinate:* $g(r\boldsymbol{x} + s\boldsymbol{y}, \boldsymbol{z}) = r^* g(\boldsymbol{x}, \boldsymbol{z}) + s^* g(\boldsymbol{y}, \boldsymbol{z})$.

2. *Linearity in second coordinate:* $g(\boldsymbol{x}, r\boldsymbol{y} + s\boldsymbol{z}) = r\, g(\boldsymbol{x}, \boldsymbol{y}) + s\, g(\boldsymbol{x}, \boldsymbol{z})$.

A sesquilinear form that satisfies

$$g(\boldsymbol{x}, \boldsymbol{y}) = g(\boldsymbol{y}, \boldsymbol{x})^*, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V},$$

is said to be Hermitian and is called a Hermitian form [29, p. 9].

Definition 4. *Let $\mathcal{B} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ be an ordered basis [127, p. 208] for a vector space $\mathcal{V}$ and let $g : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ be a sesquilinear from. The $n \times n$ matrix*

$$\eta_{\mathcal{B}} = (a_{ij}) = (g(\boldsymbol{b}_i, \boldsymbol{b}_j)), \quad i, j \in \{1, \ldots, n\}\,,$$

is called the matrix of the sesquilinear form g in the basis $\mathcal{B}$.

We take the liberty of referring to the matrix of the sesquilinear form by the term “metric”.

Consider two elements $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ expanded in terms of the ordered basis $\mathcal{B} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$; that is,

$$\boldsymbol{x}_{\mathcal{B}} = \sum_{i=1}^{n} x_i \boldsymbol{b}_i \quad \text{and} \quad \boldsymbol{y}_{\mathcal{B}} = \sum_{i=1}^{n} y_i \boldsymbol{b}_i,$$

where $x_i, y_i \in \mathbb{C}$. Then from Defn. 3 we have

$$g(\boldsymbol{x}_{\mathcal{B}}, \boldsymbol{y}_{\mathcal{B}}) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^* y_j \, g(\boldsymbol{b}_i, \boldsymbol{b}_j) = \sum_{i=1}^{n} x_i^* \left(\sum_{j=1}^{n} a_{ij} y_j \right) = \boldsymbol{x}_{\mathcal{B}}{}^{\dagger} \eta_{\mathcal{B}} \boldsymbol{y}_{\mathcal{B}}. \tag{B.5.1}$$

Hence the sesquilinear form is completely determined by the matrix $\eta_{\mathcal{B}}$, the matrix of the sesquilinear form.

The Hermiticity of g is also directly related to the Hermiticity of $\eta_{\mathcal{B}}$. Let g, as given in (B.5.1), be Hermitian. Applying Hermitian conjugation to (B.5.1), we obtain

$$g(\boldsymbol{y}_{\mathcal{B}}, \boldsymbol{x}_{\mathcal{B}})^* = \sum_{i=1}^{n} \sum_{j=1}^{n} y_i x_j^* \, g(\boldsymbol{b}_i, \boldsymbol{b}_j)^* = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^* y_j \, g(\boldsymbol{b}_j, \boldsymbol{b}_i)^*. \tag{B.5.2}$$

Comparison with (B.5.1) yields

$$g(\boldsymbol{x}_{\mathcal{B}}, \boldsymbol{y}_{\mathcal{B}}) = g(\boldsymbol{y}_{\mathcal{B}}, \boldsymbol{x}_{\mathcal{B}})^* \iff \eta_{\mathcal{B}} = \eta_{\mathcal{B}}{}^{\dagger}, \tag{B.5.3}$$

that is, the sesquilinear form g is Hermitian if and only if the matrix $\eta_{\mathcal{B}}$ is Hermitian.

Definition 5. *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. An inner product on $\mathcal{V}$ is a sesquilinear form $g : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ that is Hermitian and that satisfies the following two properties [127, p. 157] for all $\boldsymbol{x} \in \mathcal{V}$.*

1. *Positive definiteness: $g(\boldsymbol{x}, \boldsymbol{x}) \geq 0$.*
2. *Non-degeneracy: $g(\boldsymbol{x}, \boldsymbol{x}) = 0 \iff \boldsymbol{x} = 0$.*

A complex vector space $\mathcal{V}$ along with an inner product g defined on $\mathcal{V}$ is a complex inner product space $(\mathcal{V}, g)$ [127, p. 158].

Definition 6. *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. A linear functional is a mapping $\phi : \mathcal{V} \to \mathbb{C}$ that satisfies*

$$\phi(r\boldsymbol{x} + s\boldsymbol{y}) = r\,\phi(\boldsymbol{x}) + s\,\phi(\boldsymbol{y})$$

for all scalars $r, s \in \mathbb{C}$ and vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

We denote the set of all linear functionals $\phi : \mathcal{V} \to \mathbb{C}$ by $\overline{\mathcal{V}}$. It is easy to see that $\overline{\mathcal{V}}$ is itself a vector space over $\mathbb{C}$ by noting from Defn. 6 that we can add linear maps and multiply them by scalars in accordance with Defn. 1. We call $\overline{\mathcal{V}}$ the algebraic dual space of $\mathcal{V}$.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ and consider the mapping $\phi_{\boldsymbol{x}} : \mathcal{V} \to \mathbb{C}$ defined by

$$\phi_{\boldsymbol{x}}(\boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{y}), \tag{B.5.4}$$

where g is a sesquilinear form on $\mathcal{V} \times \mathcal{V}$. It immediately follows from Defn. 3 that for every $\boldsymbol{x} \in \mathcal{V}$, $\phi_{\boldsymbol{x}}$ is a linear functional on $\mathcal{V}$ in accordance with Defn. 6. We may thus define a mapping $\tau : \mathcal{V} \to \overline{\mathcal{V}}$ by

$$\tau(\boldsymbol{x}) = \phi_{\boldsymbol{x}}. \tag{B.5.5}$$

From the definition of $\overline{\mathcal{V}}$, we have

$$\phi_{\boldsymbol{x}} \in \overline{\mathcal{V}}, \quad \forall \boldsymbol{x} \in \mathcal{V}. \tag{B.5.6}$$

The basis of $\overline{\mathcal{V}}$ is now given as follows. Let $\mathcal{V}$ be a vector space with basis $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ and let g be a sesquilinear form on $\mathcal{V} \times \mathcal{V}$ with the property

$$g(\boldsymbol{b}_i, \boldsymbol{b}_j) = \delta^i{}_j, \quad \forall i, j \in \{1, \ldots, n\}, \tag{B.5.7}$$

or equivalently

$$\bar{\boldsymbol{b}}_i \boldsymbol{b}_j = \delta^i{}_j, \quad \forall i, j \in \{1, \ldots, n\}, \tag{B.5.8}$$

where $\bar{\boldsymbol{b}}_i$ is given by $\bar{\boldsymbol{b}}_i = \phi_{\boldsymbol{b}_i}$ as per (B.5.4). Then by the proof given in [129, p. 132], $\overline{\mathcal{B}} = \left\{\bar{\boldsymbol{b}}_1, \ldots, \bar{\boldsymbol{b}}_n\right\}$ is a basis for $\overline{\mathcal{V}}$. Furthermore, by [129, Thm. 6, VII, §4], $\dim(\mathcal{V}) = \dim(\overline{\mathcal{V}})$, and thus, by [127, Thm. 2.4], $\mathcal{V}$ is isomorphic to $\overline{\mathcal{V}}$.

For the purposes of application it will be most convenient to explore (B.5.4) and (B.5.5) in terms of an ordered basis for $\mathcal{V}$ as in Defn. 4 above. For the linear functional (B.5.4) we have

$$\phi_{\boldsymbol{x}_\mathcal{B}}(\boldsymbol{y}_\mathcal{B}) = g(\boldsymbol{x}_\mathcal{B}, \boldsymbol{y}_\mathcal{B}) = {\boldsymbol{x}_\mathcal{B}}^\dagger \eta_\mathcal{B} \boldsymbol{y}_\mathcal{B}. \tag{B.5.9}$$

Accordingly the mapping $\tau : \mathcal{V} \to \overline{\mathcal{V}}$ is given by

$$\tau(\boldsymbol{x}_\mathcal{B}) = {\boldsymbol{x}_\mathcal{B}}^\dagger \eta_\mathcal{B}. \tag{B.5.10}$$

It is thus apparent that τ is conjugate linear.

Moreover if g is chosen as per (B.5.7) the basis for $\overline{\mathcal{V}}$ is given by

$$\overline{\mathcal{B}} = \left\{ {\boldsymbol{b}_i}^\dagger \eta_\mathcal{B} \mid \boldsymbol{b}_i \in \mathcal{B} \right\}. \tag{B.5.11}$$

B.5.2 G-invariance of the sesquilinear form

We will now explore the G-invariance, that is the invariance under the action of a group G, of a sesquilinear form. Of course, the group in question is the restricted Lorentz group $\mathcal{L}_+^\uparrow$.

Consider a vector space $\mathcal{V}$ with basis $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ along with a sesquilinear form $g : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ determined by a matrix $\eta_\mathcal{B}$, the matrix of the sesquilinear form as per (B.5.1). We will omit the subscripts for the remainder of the section. For a representation Ω of $\mathcal{L}_+^\uparrow$ on the $\mathcal{V}$, the demand of G-invariance in terms of the basis vectors then reads

$${\boldsymbol{b}_i}^\dagger \eta\, \boldsymbol{b}_j = (\Omega \boldsymbol{b}_i)^\dagger \eta\, \Omega \boldsymbol{b}_j, \quad \forall\, \boldsymbol{b}_i, \boldsymbol{b}_j \in \mathcal{B}. \tag{B.5.12}$$

Of course, $(\Omega \boldsymbol{b}_i)^\dagger = {\boldsymbol{b}_i}^\dagger \Omega^\dagger$; hence, (B.5.12) will be satisfied if η obeys the relation[2]

$$\Omega^\dagger \eta\, \Omega = \eta. \tag{B.5.13}$$

[2] In the case where the components of Ω are real, this is equivalent to (2.1.3) in matrix form.

This can be written in terms of the generators of the Lie algebra of the restricted Lorentz group. For Ω a Lorentz boost, we have $\Omega_j = \exp[i\, K_j \varphi_j]$ where K_j are the three generators of Lorentz boost and φ_j are the corresponding parameters. Expanding $\exp[i\, K_j \varphi_j]$ infinitesimally to first order in φ_j and substituting for Ω in (B.5.13) we obtain

$$K_i{}^\dagger \eta = \eta K_i, \quad \forall i \in \{x, y, z\}. \tag{B.5.14}$$

For Ω a rotation, we have $\Omega_j = \exp[i\, J_j \theta_j]$ where J_j are the rotation generators and θ_j are the corresponding parameters. Expanding $\exp[i\, J_j \theta_j]$ infinitesimally to first order in θ_j and substituting for Ω in (B.5.13) we obtain

$$J_i{}^\dagger \eta = \eta J_i, \quad \forall i \in \{x, y, z\}. \tag{B.5.15}$$

In the case of K_i anti-Hermitian and J_i Hermitian, (B.5.14) and (B.5.15) give the following constraints on the matrix of the sesquilinear form:

$$\{K_i, \eta\} = 0 \quad \text{and} \quad [J_i, \eta] = 0, \quad \forall i \in \{x, y, z\}\,. \tag{B.5.16}$$

C

Explicit expansions of boost and rotation operators

We here provide the Maclaurin series expansions of various representations of rotation and boost that are used throughout the text.

C.1 Rotation operator for $j = 1/2$ or $j = 1$

The rotation operator is given by $\exp[i\,\boldsymbol{J}\cdot\boldsymbol{\theta}]$ where

$$\boldsymbol{J} \equiv (J_x, J_y, J_z) \quad \text{and} \quad \boldsymbol{\theta} \equiv (\theta_x, \theta_y, \theta_z) = \theta\,\hat{\boldsymbol{\theta}}. \tag{C.1.1}$$

We shall refer to $\boldsymbol{\theta}$ as the rotation vector; its components θ_i, $i \in \{x, y, z\}$, are the angles of rotation about the axes of Cartesian three-space. The corresponding rotation generators are J_i. We here consider the case where the following identity holds:

$$\left(i\,\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}}\right) = -\left(i\,\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}}\right)^3. \tag{C.1.2}$$

The Maclaurin series expansion thus reads

$$\begin{aligned}
&\exp[i\boldsymbol{J}\cdot\boldsymbol{\theta}] \\
&\quad = \mathbb{1} + i\boldsymbol{J}\cdot\boldsymbol{\theta} + \frac{1}{2}(i\boldsymbol{J}\cdot\boldsymbol{\theta})^2 + \frac{1}{3!}(i\boldsymbol{J}\cdot\boldsymbol{\theta})^3 + \frac{1}{4!}(i\boldsymbol{J}\cdot\boldsymbol{\theta})^4 + \dots \\
&\quad = \mathbb{1} + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\theta + \frac{1}{2}(i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2\theta^2 + \frac{1}{3!}(i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^3\theta^3 + \frac{1}{4!}(i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^4\theta^4 + \dots \\
&\quad = \mathbb{1} + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\left[\theta - \frac{1}{3!}\theta^3 + \dots\right] + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2\left[\frac{1}{2}\theta^2 - \frac{1}{4!}\theta^4 + \dots\right] \\
&\quad = \mathbb{1} + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\left[\sin(\theta)\right] + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2\left[1 - \cos(\theta)\right],
\end{aligned} \tag{C.1.3}$$

where $\mathbb{1}$ is an identity matrix of the appropriate dimensions. For any given representation, the generators of which satisfy (C.1.2), the explicit form of the rotation operators $R_x(\theta)$, $R_y(\theta)$, and $R_z(\theta)$ about the axes of Cartesian three-space are readily obtained from (C.1.3) by an appropriate choice of $\hat{\boldsymbol{\theta}}$.

C.2 Boost operator for $j = 1/2$ or $j = 1$

The boost operator is given by $\exp[i\,\boldsymbol{K}\cdot\boldsymbol{\varphi}]$ where

$$\boldsymbol{K} = (K_x, K_y, K_z) \quad \text{and} \quad \boldsymbol{\varphi} = (\varphi_x, \varphi_y, \varphi_z) = \varphi\,\hat{\boldsymbol{\varphi}}. \tag{C.2.1}$$

We shall refer to $\boldsymbol{\varphi}$ as the rapidity vector. It is chosen to be aligned with the three momentum; hence, $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{p}}$ where $\hat{\boldsymbol{p}} \equiv \boldsymbol{p}/|\boldsymbol{p}|$. The magnitude φ of the rapidity vector is called the rapidity parameter. A physical interpretation is given by the parametrisation (C.2.4).

We here consider the case where the following identity holds:

$$(i\,\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}}) = (i\,\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}})^3\,. \tag{C.2.2}$$

The Maclaurin series expansion thus reads

$$\begin{aligned}
&\exp[i\boldsymbol{K}\cdot\boldsymbol{\varphi}] \\
&\quad = \mathbb{1} + i\boldsymbol{K}\cdot\boldsymbol{\varphi} + \frac{1}{2}(i\boldsymbol{K}\cdot\boldsymbol{\varphi})^2 + \frac{1}{3!}(i\boldsymbol{K}\cdot\boldsymbol{\varphi})^3 + \frac{1}{4!}(i\boldsymbol{K}\cdot\boldsymbol{\varphi})^4 + \ldots \\
&\quad = \mathbb{1} + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\varphi + \frac{1}{2}(i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\varphi^2 + \frac{1}{3!}(i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^3\varphi^3 + \frac{1}{4!}(i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^4\varphi^4 + \ldots \\
&\quad = \mathbb{1} + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\left[\varphi + \frac{1}{3!}\varphi^3 + \ldots\right] + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\left[\frac{1}{2}\varphi^2 + \frac{1}{4!}\varphi^4 + \ldots\right] \\
&\quad = \mathbb{1} + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\,[\sinh(\varphi)] + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\,[\cosh(\varphi) - 1]\,.
\end{aligned} \tag{C.2.3}$$

where again $\mathbb{1}$ is an identity matrix of the appropriate dimension; φ is parameterised by

$$\cosh(\varphi) = \frac{E}{m} \quad \text{and} \quad \sinh(\varphi) = \frac{|\boldsymbol{p}|}{m}, \tag{C.2.4}$$

where $E = \sqrt{|\boldsymbol{p}|^2 + m^2}$ is the relativistic energy and $|\boldsymbol{p}|$ is the magnitude of the three-momentum. These are, of course, related to the velocity of a moving particle and to its rest mass by $E = \gamma m$ and $|\boldsymbol{p}| = \gamma m|\boldsymbol{v}|$, where $\gamma = (1-\beta^2)^{-1/2}$. An equivalent parametrisation to (C.2.4) is thus given by

$$\cosh(\varphi) = \gamma \quad \text{and} \quad \sinh(\varphi) = \gamma\beta, \tag{C.2.5}$$

where $\beta = |\boldsymbol{v}|$ in the units $c = 1$. One might convince oneself of the validity of this identification by recalling the trigonometric and algebraic relations

$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= 1, \\
\frac{E^2}{m^2} \quad - \quad \frac{p^2}{m^2} \quad &= 1, \\
\gamma^2 \quad - \quad (\beta\gamma)^2 \quad &= 1,
\end{aligned}$$

and noting that the domains and codomains of the functions on either side of the equals signs in (C.2.4) and (C.2.5) match. Furthermore, although one clearly has the freedom to scale φ at will, this must occur in a consistent fashion in all formulations of the Lorentz boost regardless of the particular representation in question. Choosing $x = \varphi$ in one place and $x = \varphi/2$ in another would render the physical interpretation of the rapidity parameter meaningless.

C.3 The $(1/2, 0)$ representation

C.3.1 Rotation

The spin one-half rotation generators, as obtained from (2.3.30) by choosing $s = 1/2$, read

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.3.1}$$

Now consider the matrix

$$2i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}} = \frac{i}{\theta}\begin{pmatrix} \theta_z & \theta_x - ip_y \\ \theta_x + i\theta_y & -\theta_z \end{pmatrix}. \tag{C.3.2}$$

Squaring this matrix, we find $(2i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2 = -\mathbb{1}_2$; hence, the identity (C.1.2) is satisfied and we obtain from (C.1.3)

$$\begin{aligned} \exp[i\boldsymbol{J}\cdot\boldsymbol{\theta}] &= \mathbb{1}_2 + (2i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\,[\sin(\theta/2)] + (2i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2\,[1-\cos(\theta/2)] \\ &= \mathbb{1}_2\cos(\theta/2) + (2i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\sin(\theta/2). \end{aligned} \tag{C.3.3}$$

C.3.2 Boost

The three generators of Lorentz boost in the $(1/2, 0)$ representation are given in terms of the corresponding rotation generators by $K_i = -iJ_i$. We thus have

$$K_x = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K_y = \frac{1}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad K_z = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.3.4}$$

Now consider the matrix

$$2i\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}} = 2i\boldsymbol{K}\cdot\hat{\boldsymbol{p}} = \frac{1}{p}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}. \tag{C.3.5}$$

Computing the square of (C.3.5), we find that $(2i\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}})^2 = \mathbb{1}_2$; hence, the identity (C.2.2) is satisfied and we obtain from (C.2.3)

$$\begin{aligned} \exp[i\boldsymbol{K}\cdot\boldsymbol{\varphi}] &= \mathbb{1}_2 + (2i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\,[\sinh(\varphi/2)] + (2i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\,[\cosh(\varphi/2)-1] \\ &= \mathbb{1}_2\cosh(\varphi/2) + (2i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\sinh(\varphi/2). \end{aligned} \tag{C.3.6}$$

Before we insert (C.2.4), we must first rewrite (C.3.6) in terms of $\cosh(\varphi)$ and $\sinh(\varphi)$. The relevant half angle formulae read

$$\sinh(x/2) = \mathrm{sgn}(x)\sqrt{\frac{\cosh(x)-1}{2}} \quad \text{and} \quad \cosh(x/2) = \sqrt{\frac{\cosh(x)+1}{2}}, \tag{C.3.7}$$

for all $x \in \mathbb{R}$. Furthermore, considering that $\cosh(\varphi) = E/m$ where $E/m \geq 1$, we have $\mathrm{sgn}(\varphi) = 1$. The boost expansion thus becomes

$$\begin{aligned} \exp[i\boldsymbol{K}\cdot\boldsymbol{\varphi}] &= \mathbb{1}_2\sqrt{\frac{E/m+1}{2}} + (2i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\sqrt{\frac{E/m-1}{2}} \\ &= \mathbb{1}_2\sqrt{\frac{E+m}{2m}} + (2i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\sqrt{\frac{E-m}{2m}} \\ &= \sqrt{\frac{E+m}{2m}}\left[\mathbb{1}_2 + (2i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\sqrt{\frac{E-m}{E+m}}\right] \\ &= \sqrt{\frac{E+m}{2m}}\left[\mathbb{1}_2 + (2i\boldsymbol{K}\cdot\boldsymbol{p})\frac{1}{E+m}\right]. \end{aligned} \tag{C.3.8}$$

C.4 The $(1/2, 0) \oplus (0, 1/2)$ representation

C.4.1 Rotation

The rotation operator of the $(1/2, 0) \oplus (0, 1/2)$ representation is given by the direct sum of the respective rotation operators of $(1/2, 0)$ and $(0, 1/2)$. These are identical. We may thus use the result obtained in the App. C.3.1 to write

$$\exp[i\boldsymbol{J}\cdot\boldsymbol{\theta}] = \mathbb{1}_4\cos(\theta/2) + (2i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\sin(\theta/2), \tag{C.4.1}$$

where the rotation generators are given by

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$
$$J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{C.4.2}$$

C.4.2 Boost

The boost operator of the $(1/2, 0) \oplus (0, 1/2)$ representation is given by the direct sum of the respective boost operators of $(1/2, 0)$ and $(0, 1/2)$. These differ only by a sign: for

$(1/2,0)$ we have $K_i = -iJ_i$, whereas for $(0,1/2)$ we have $K_i = +iJ_i$. We may thus use the result obtained in the App. C.3.2 to write

$$\exp[i\boldsymbol{K}\cdot\boldsymbol{\varphi}] = \sqrt{\frac{E+m}{2m}}\left[\mathbb{1}_4 + (2i\boldsymbol{K}\cdot\boldsymbol{p})\,\frac{1}{E+m}\right], \tag{C.4.3}$$

where the boost generators are given by

$$K_x = \frac{1}{2i}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad K_y = \frac{1}{2i}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},$$

$$K_z = \frac{1}{2i}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{C.4.4}$$

C.5 The $(1,0)$ representation

C.5.1 Rotation

The spin one rotation generators, as obtained from (2.3.30) by choosing $s=1$, read

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{C.5.1}$$

Consider the matrix

$$i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}} = \frac{i}{\theta\sqrt{2}}\begin{pmatrix} \theta_z\sqrt{2} & \theta_x - i\theta_y & 0 \\ \theta_x + i\theta_y & 0 & \theta_x - i\theta_y \\ 0 & \theta_x + i\theta_y & -\theta_z\sqrt{2} \end{pmatrix}. \tag{C.5.2}$$

Computing the cube of (C.5.2), we find that the identity (C.1.2) is satisfied. We thus obtain from (C.1.3)

$$\exp[i\boldsymbol{J}\cdot\boldsymbol{\theta}] = \mathbb{1}_3 + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\,[\sin(\theta)] + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2\,[1-\cos(\theta)]\,. \tag{C.5.3}$$

An alternate representation to the above is given by

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{C.5.4}$$

This is called the adjoint representation.

C.5.2 Boost

The three generators of Lorentz boost in the $(1,0)$ representation are given in terms of the corresponding rotation generators by $K_i = -iJ_i$. Using the generators given in (C.5.1), we obtain

$$K_x = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad K_y = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad K_z = \frac{1}{i}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{C.5.5}$$

Consider the matrix

$$i\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}} = i\boldsymbol{K}\cdot\hat{\boldsymbol{p}} = \frac{1}{p\sqrt{2}}\begin{pmatrix} p_z\sqrt{2} & p_x - i\theta_y & 0 \\ p_x + ip_y & 0 & p_x - ip_y \\ 0 & p_x + ip_y & -p_z\sqrt{2} \end{pmatrix}. \tag{C.5.6}$$

Computing the cube of (C.5.6), we find that the identity (C.2.2) is satisfied. We thus obtain from (C.2.3)

$$\exp[i\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}}] = \mathbb{1}_3 + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\,[\sinh(\varphi)] + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\,[\cosh(\varphi)-1]\,. \tag{C.5.7}$$

Inserting (C.2.4), and noting that

$$\frac{E}{m} - 1 = \frac{1}{m}\,[E-m] = \frac{1}{m}\left[\frac{E^2-m^2}{E+m}\right] = \frac{p^2}{m\,(E+m)}, \tag{C.5.8}$$

we obtain

$$\exp[i\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}}] = \mathbb{1}_3 + (i\boldsymbol{K}\cdot\boldsymbol{p})\frac{1}{m} + (i\boldsymbol{K}\cdot\boldsymbol{p})^2\frac{1}{m\,(E+m)}. \tag{C.5.9}$$

C.6 Boost operator for $(1,0)\oplus(0,1)$

C.6.1 Rotation

The rotation operator of the $(1,0)\oplus(0,1)$ representation is given by the direct sum of the respective rotation operators of $(1,0)$ and $(0,1)$. These are identical. We may thus use the result obtained in the App. C.5.1 to write

$$\exp[i\boldsymbol{J}\cdot\boldsymbol{\theta}] = \mathbb{1}_6 + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\,[\sin(\theta)] + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2\,[1-\cos(\theta)]\,, \tag{C.6.1}$$

where the rotation generators are given by

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \end{pmatrix},$$

$$J_z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{C.6.2}$$

C.6.2 Boost

The boost operator of the $(1,0) \oplus (0,1)$ representation is given by the direct sum of the respective boost operators of $(1,0)$ and $(0,1)$. These differ only by a sign: for $(1,0)$ we have $K_i = -iJ_i$, whereas for $(0,1)$ we have $K_i = +iJ_i$. We may thus use the result obtained in the App. C.5.2 to write

$$\exp[i\boldsymbol{K} \cdot \hat{\varphi}] = \mathbb{1}_6 + (i\boldsymbol{K} \cdot \boldsymbol{p})\frac{1}{m} + (i\boldsymbol{K} \cdot \boldsymbol{p})^2 \frac{1}{m\,(E+m)}, \tag{C.6.3}$$

where the boost generators are given by

$$K_x = \frac{1}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad K_y = \frac{1}{i\sqrt{2}} \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & i \\ 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix},$$

$$K_z = \frac{1}{i} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{C.6.4}$$

C.7 The $(1/2, 1/2)$ representation

The operators of the $(1/2, 1/2)$ representation are given by the tensor product of the corresponding operators of the representation $(1/2, 0)$ with those of the representation $(0, 1/2)$. To avoid the algebraic tedium that would ensue if we were to attempt to compute the tensor product of the closed forms given in App. C.3, we will instead take the tensor product of the respective infinitesimal transformations. Once these are obtained, we can proceed in the same manner as for the other cases considered above to construct the desired operators.

Be begin by recalling the Pauli spin matrices. These read

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.7.1}$$

Of course, the generators of the $(1/2, 0)$ are then given by $J_j = \sigma_j/2$ and $K_j = \sigma_j/2i$, whereas for $(0, 1/2)$ we have $J_j = \sigma_j/2$ and $K_j = -\sigma_j/2i$.

C.7.1 Rotation

Consider an infinitesimal rotation of the representation $(1/2, 0)$ about the x-axis

$$(\boldsymbol{r}_x)_{ij} = \delta^i{}_j + i\,\epsilon\,(\sigma_x/2)_{ij}, \tag{C.7.2}$$

where $i, j \in \{1, 2\}$, ϵ is an infinitesimal rotation parameter. As follows immediately from the discussion in App. C.4, the counterpart of (C.7.2) in the $(0, 1/2)$ representation is identical. Hence, from the definition of the tensor product, the corresponding infinitesimal rotation in the $(1/2, 1/2)$ representation reads

$$\mathcal{R}_x = \begin{pmatrix} \boldsymbol{r}_{x11}\boldsymbol{r}_x & \boldsymbol{r}_{x12}\boldsymbol{r}_x \\ \boldsymbol{r}_{x21}\boldsymbol{r}_x & \boldsymbol{r}_{x22}\boldsymbol{r}_x \end{pmatrix}. \tag{C.7.3}$$

Expanding this in terms of (C.7.2) yields the following generator of rotation about the x-axis in the $(1/2, 1/2)$ representation:

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{C.7.4}$$

Repeating this procedure for the two remaining rotation operators, one obtains

$$J_y = \frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix} \quad \text{and} \quad J_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{C.7.5}$$

Having thus derived the generators we can proceed to compute the corresponding operators. Consider the matrix

$$i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}} = \frac{i}{\theta\,2}\begin{pmatrix} 2\,\theta_z & \theta_x - i\theta_y & \theta_x - i\theta_y & 0 \\ \theta_x + i\theta_y & 0 & 0 & \theta_x - i\theta_y \\ \theta_x + i\theta_y & 0 & 0 & \theta_x - i\theta_y \\ 0 & \theta_x + i\theta_y & \theta_x + i\theta_y & -2\,\theta_z \end{pmatrix}. \tag{C.7.6}$$

Computing the cube of (C.7.6), we find that the identity (C.1.2) is satisfied. We thus obtain from (C.1.3)

$$\exp[i\boldsymbol{J}\cdot\boldsymbol{\theta}] = \mathbb{1}_4 + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})\,[\sin(\theta)] + (i\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}})^2\,[1-\cos(\theta)]\,. \tag{C.7.7}$$

C.7.2 Boost

To derive the boost operator of the $(1/2, 1/2)$ representation, we again begin by deriving the underlying generators. Consider the infinitesimal boost operators

$$(\boldsymbol{b}_x^R)_{ij} = \delta^i{}_j + i\,\epsilon\,(-\,i\,\sigma_x/2)_{ij}, \quad (\boldsymbol{b}_x^L)_{ij} = \delta^i{}_j + i\,\epsilon\,(+\,i\,\sigma_x/2)_{ij}. \tag{C.7.8}$$

Here $\boldsymbol{b}_x^R$ is an infinitesimal boost of the representation $(1/2, 0)$ along the x-axis; $\boldsymbol{b}_x^L$ is the corresponding infinitesimal boost of the representation $(0, 1/2)$. The indices $ij \in \{1, 2\}$ and ϵ is an infinitesimal boost parameter. Computing the tensor product $\boldsymbol{b}_x^L \otimes \boldsymbol{b}_x^R$, we obtain

$$\mathcal{B}_x = \begin{pmatrix} \boldsymbol{b}^R_{x11}\boldsymbol{b}^L_x & \boldsymbol{b}^R_{x12}\boldsymbol{b}^L_x \\ \boldsymbol{b}^R_{x21}\boldsymbol{b}^L_x & \boldsymbol{b}^R_{x22}\boldsymbol{b}^L_x \end{pmatrix}, \tag{C.7.9}$$

an infinitesimal boost along the x-axis in the $(1/2, 1/2)$ representation. Expanding (C.7.9) using (C.7.8) yields the following generator of boost along the x-axis:

$$K_x = \frac{1}{2}\begin{pmatrix} 0 & i & -i & 0 \\ i & 0 & 0 & -i \\ -i & 0 & 0 & i \\ 0 & -i & i & 0 \end{pmatrix}. \tag{C.7.10}$$

Repeating this procedure for the two remaining boost operators, one obtains

$$K_y = \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.7.11}$$

Having thus obtained the infinitesimal generators we can proceed to compute the corresponding operators. Consider the matrix

$$i\boldsymbol{K}\cdot\hat{\boldsymbol{\varphi}} = i\boldsymbol{K}\cdot\hat{\boldsymbol{p}} = \frac{1}{p2}\begin{pmatrix} 0 & -p_x+ip_y & p_x-ip_y & 0 \\ -p_x-ip_y & 2\,p_z & 0 & p_x-ip_y \\ p_x+ip_y & 0 & -2\,p_z & -p_x+ip_y \\ 0 & p_x+ip_y & -p_x-ip_y & -p_z \end{pmatrix}. \tag{C.7.12}$$

Computing the cube of (C.7.12), we find that the identity (C.2.2) is satisfied. We thus obtain from (C.2.3)

$$\exp[i\boldsymbol{K}\cdot\boldsymbol{\varphi}] = \mathbb{1}_4 + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})\,[\sinh(\varphi)] + (i\boldsymbol{K}\cdot\hat{\boldsymbol{p}})^2\,[\cosh(\varphi)-1]\,. \tag{C.7.13}$$

This, of course, can be rewritten as

$$\exp[i\boldsymbol{K}\cdot\boldsymbol{\varphi}] = \mathbb{1}_4 + (i\boldsymbol{K}\cdot\boldsymbol{p})\frac{1}{m} + (i\boldsymbol{K}\cdot\boldsymbol{p})^2\frac{1}{m\,(E+m)} \tag{C.7.14}$$

via the use of (C.2.4) and (C.5.8).

D
Publications

This appendix contains two publications that are based upon research undertaken by the author of the present work and his colleagues on the Lorentz violating dark matter candidate Elko. They were published in Physics Letters B and Physical Review D, respectively.

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Elko as self-interacting fermionic dark matter with axis of locality

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ABSTRACT

We here provide further details on the construction and properties of mass dimension one quantum fields based on Elko expansion coefficients. We show that by a judicious choice of phases, the locality structure can be dramatically improved. In the process we construct a fermionic dark matter candidate which carries not only an unsuppressed quartic self interaction but also a preferred axis. Both of these aspects are tentatively supported by the data on dark matter.

1. Introduction

If one wishes to treat Majorana spinors in their own right as four-component spinors, and not as Weyl spinors in disguise (or, as G-numbers), one must extend them in such a way that not only the $+1$ eigenvalue, under charge conjugation operator, but also the -1 eigenvalue is incorporated. This was the starting point of the Elko formalism, and the unexpected results, reported in Refs. [1,2]. It was recognised by the authors of these papers that the usual introduction of a Majorana mass term still leaves a problem with the free Lagrangian density, and that to prevent the Dirac-type mass term from vanishing identically, one had to invoke a new dual. The mentioned problem is akin to the one mentioned by Aitchison and Hey [3, Appendix P]. However, the authors of the Elko formalism chose not to follow the Grassmannisation of the Majorana spinors. It is in this departure that several new results were obtained. Most unexpected of these was the mass dimensionality of the field.

The new dual appeared as an ad hoc construct in the mentioned works. Here we give a full justification for the introduction of the Elko dual. Similarly, the locality structure investigated in the original papers failed to fully appreciate the necessity of certain phases in the expansion coefficients in a field operator.[1] Here we attend to that and learn of their dramatic effects on the locality structure.

At present, the quartic self interaction, as well as a preferred axis in the dark sector, are observationally favoured for dark matter candidates [6–12]. In this communication we provide an *ab initio* evidence that both of these aspects are naturally present in the Elko dark matter.

To avoid confusion, we note that spinors of the Elko formalism have spawned an intense activity among a group of mathematical physicists and cosmologists [13–27]. Similar to the work of Gillard and Martin [28] the emphasis in this communication is on the quantum fields, and not so much on the spinors.

2. Theory of self-interacting fermionic dark matter with axis of locality

In this section we outline the construction of two quantum fields with Elko as expansion coefficients. The full details shall appear in an archival paper elsewhere.

2.1. Notation

Let $\phi(\mathbf{p})$ be a left-handed (ℓ) Weyl spinor of spin one half. Under a Lorentz boost, it transforms as $\phi(\mathbf{p}) = \kappa_\ell \phi(\mathbf{0})$ where[2]

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[1] The authors of the original Elko papers are not be too harshly criticised for these lapses as almost every textbook on quantum field suffers from a similar neglect. Two notable exceptions are the recent classics by Weinberg [4] and Srednicki [5]. The authors of the present communication acknowledge the insights gained from these monographs.

[2] The boost parameter $\boldsymbol{\varphi} = \varphi\hat{\mathbf{p}}$, in terms of energy E and momentum $\mathbf{p} = p\hat{\mathbf{p}}$ associated with a particle of mass m, is given by $\cosh(\boldsymbol{\varphi}) = E/m$ and $\sinh(\boldsymbol{\varphi}) =$

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$$\kappa_\ell = \exp\left(-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right) = \varrho\left(\mathbb{I} - \beta^{-1}\boldsymbol{\sigma}\cdot\mathbf{p}\right), \tag{1}$$

with

$$\varrho := \sqrt{\frac{E+m}{2m}} \quad \text{and} \quad \beta := E+m. \tag{2}$$

Here, the $\mathbf{0}$ is to be interpreted as $\mathbf{p}|_{p\to 0}$, and not as $\mathbf{p}|_{p=0}$. This restriction can be removed, if necessary (for example, by working in 'polarisation basis' which then comes with its own subtleties). We choose $\phi(\mathbf{p})$ to belong to one of the two possible helicities: $\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\phi_\pm(\mathbf{p}) = \pm\phi_\pm(\mathbf{p})$. Following Ref. [2] note that, (a) under a Lorentz boost, $\vartheta\Theta\phi^*(\mathbf{p})$ transforms as a right-handed (r) Weyl spinor, $[\vartheta\Theta\phi^*(\mathbf{p})] = \kappa_r[\vartheta\Theta\phi^*(\mathbf{0})]$, with

$$\kappa_r = \exp\left(+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right) = \varrho\left(\mathbb{I} + \beta^{-1}\boldsymbol{\sigma}\cdot\mathbf{p}\right), \tag{3}$$

where ϑ is an unspecified phase to be determined below, and Θ is Wigner's time reversal operator for spin one half, $\Theta[\boldsymbol{\sigma}/2]\Theta^{-1} = -[\boldsymbol{\sigma}/2]^*$; and (b) the helicity of $\vartheta\Theta\phi^*(\mathbf{p})$ is *opposite* to that of $\phi(\mathbf{p})$,

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\left[\vartheta\Theta\phi^*_\pm(\mathbf{p})\right] = \mp\left[\vartheta\Theta\phi^*_\pm(\mathbf{p})\right]. \tag{4}$$

In terms of $\Theta(= -i\sigma_2)$, the charge conjugation operator in the $r\oplus\ell$ spinorial space reads

$$S(C) = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K, \tag{5}$$

where K is the complex conjugation operator.

2.2. Elko

Elko abbreviates the German phrase **E**igenspinoren des **L**adungs**k**onjugations**o**perators. The four-component *dual* helicity spinors

$$\chi(\mathbf{p}) = \begin{pmatrix} \vartheta\Theta\phi^*(\mathbf{p}) \\ \phi(\mathbf{p}) \end{pmatrix}, \tag{6}$$

become eigenspinors of the charge conjugation operator, i.e. Elko, with eigenvalues ± 1 if the phase ϑ is set to $\pm i$

$$S(C)\chi(\mathbf{p})|_{\vartheta=\pm i} = \pm\chi(\mathbf{p})|_{\vartheta=\pm i}. \tag{7}$$

We parameterise a unit vector along the momentum of a particle, $\hat{\mathbf{p}}$, as $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and adopt phases so that at rest

$$\phi_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}, \tag{8}$$

$$\phi_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix}. \tag{9}$$

Eqs. (8)–(9), when coupled with Eq. (6), allow us to explicitly introduce the self-conjugate spinors ($\vartheta = +i$) and anti self-conjugate spinors ($\vartheta = -i$) at rest

$$\xi_{\{-,+\}}(\mathbf{0}) := +\chi(\mathbf{0})|_{\phi(\mathbf{0})\to\phi_+(\mathbf{0}),\,\vartheta=+i}, \tag{10}$$

$$\xi_{\{+,-\}}(\mathbf{0}) := +\chi(\mathbf{0})|_{\phi(\mathbf{0})\to\phi_-(\mathbf{0}),\,\vartheta=+i}, \tag{11}$$

$$\zeta_{\{-,+\}}(\mathbf{0}) := +\chi(\mathbf{0})|_{\phi(\mathbf{0})\to\phi_-(\mathbf{0}),\,\vartheta=-i}, \tag{12}$$

$$\zeta_{\{+,-\}}(\mathbf{0}) := -\chi(\mathbf{0})|_{\phi(\mathbf{0})\to\phi_+(\mathbf{0}),\,\vartheta=-i}. \tag{13}$$

p/m. By $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ we denote the Pauli matrices. The symbol $\mathbb{I}$ represents an identity matrix, while $\mathbb{O}$ stands for a null matrix. Their dimensionality shall be apparent from the context.

Table 1
The values of $[e_\imath(\mathbf{p})]^\dagger\eta e_\jmath(\mathbf{p})$ evaluated using η. The $\imath$ runs from 1 to 4 along the rows and $\jmath$ does the same across the columns.

0	$-im(a+b)$	$-im(a-b)$	0
$im(a+b)$	0	0	$-im(a-b)$
$-im(a-b)$	0	0	$im(a+b)$
0	$-im(a-b)$	$-im(a+b)$	0

The $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ for an arbitrary momentum are now readily obtained[3]

$$\xi(\mathbf{p}) = \kappa\xi(\mathbf{0}), \qquad \zeta(\mathbf{p}) = \kappa\zeta(\mathbf{0}), \tag{14}$$

where $\kappa := \kappa_r \oplus \kappa_\ell$. The choice of phases and the dual-helicity designations are different from those adopted in Refs. [1,2]. These changes were inspired by the considerations presented in Section 38 of Ref. [5], and by those given in Section 5.5 of Ref. [4]. These differences are crucial to the results here presented.

2.3. Elko dual

If one now invokes the Dirac dual for the ξ and ζ spinors one immediately encounters a problem in constructing a Lagrangian description [3, Appendix P.1]. This was one of the reasons that a new dual was introduced in the original papers on Elko. That dual translates to the following definition

$$\overset{\neg}{e}_{\{\mp,\pm\}}(\mathbf{p}) := \mp i\left[e_{\{\pm,\mp\}}(\mathbf{p})\right]^\dagger\gamma^0. \tag{15}$$

Its essential uniqueness can be established by looking for a 'metric' η such that the product $[e_\imath(\mathbf{p})]^\dagger\eta e_\jmath(\mathbf{p})$ — with $e_\imath(p)$ as any one of the four Elko — remains invariant under an arbitrary Lorentz transformation. This requirement can be readily shown to translate into the following constraints on η

$$[J_i, \eta] = 0, \qquad \{K_i, \eta\} = 0. \tag{16}$$

Since the only property of the generators of rotations and boosts that enters the derivation of the above constraints is that $\mathbf{J}^\dagger = \mathbf{J}$ and $\mathbf{K}^\dagger = -\mathbf{K}$, the result applies to all *finite*-dimensional representations of the Lorentz group. It need not be restricted to Elko alone. Seen in this light, there is no non-trivial solution for η either for the r-type or the ℓ-type Weyl spinors. For $r\oplus\ell$ representation space, the most general solution is found to have the form

$$\eta = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}. \tag{17}$$

It is now convenient to introduce the notation $e_1(\mathbf{p}) := \xi_{\{-,+\}}(\mathbf{p})$, $e_2(\mathbf{p}) := \xi_{\{+,-\}}(\mathbf{p})$, $e_3(\mathbf{p}) := \zeta_{\{-,+\}}(\mathbf{p})$, and $e_4(\mathbf{p}) := \zeta_{\{+,-\}}(\mathbf{p})$. Sixteen values of $[e_\imath(\mathbf{p})]^\dagger\eta e_\jmath(\mathbf{p})$ as $\imath$ and $\jmath$ vary from 1 to 4 are presented in Table 1.

To treat the r and ℓ Weyl spaces on the same footing, we set $b = a$. To make the invariant norms real, we give a and b the common value of $\pm i$; resulting in $\eta = \pm i\gamma^0$. Within the stated caveats, the uniqueness of the Elko dual, defined in Eq. (15), is now apparent.

2.4. Elko orthonormality and completeness relations

Under the new dual, the orthonormality relations read

[3] The boost operator commutes with the charge conjugation operator and for that reason $S(C)\chi(\mathbf{0}) = \pm\chi(\mathbf{0})$ implies $S(C)\chi(\mathbf{p}) = \pm\chi(\mathbf{p})$.

$$\overset{\neg}{\xi}_\alpha(\mathbf{p})\xi_{\alpha'}(\mathbf{p}) = +2m\delta_{\alpha\alpha'}, \tag{18}$$

$$\overset{\neg}{\zeta}_\alpha(\mathbf{p})\zeta_{\alpha'}(\mathbf{p}) = -2m\delta_{\alpha\alpha'}, \tag{19}$$

along with $\overset{\neg}{\xi}_\alpha(\mathbf{p})\zeta_{\alpha'}(\mathbf{p}) = 0$, and $\overset{\neg}{\zeta}_\alpha(\mathbf{p})\xi_{\alpha'}(\mathbf{p}) = 0$. The dual helicity index α ranges over the two possibilities: $\{+,-\}$ and $\{-,+\}$, and $-\{\pm,\mp\} := \{\mp,\pm\}$. The completeness relation

$$\frac{1}{2m}\sum_\alpha\left[\xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p})\right] = \mathbb{I} \tag{20}$$

establishes that we need to use *both* the self-conjugate as well as the anti self-conjugate spinors to fully capture the relevant degrees of freedom.

2.5. Elko spin sums and a preferred axis

The existence of a preferred axis, which we will later identify as the axis of locality in the dark sector, is hidden in the spin sums that appear in Eq. (20). It becomes manifest in the results:

$$\sum_\alpha \xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p}) = m\left[\mathcal{G}(\mathbf{p}) + \mathbb{I}\right], \tag{21}$$

$$\sum_\alpha \zeta_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p}) = m\left[\mathcal{G}(\mathbf{p}) - \mathbb{I}\right], \tag{22}$$

which together *define* $\mathcal{G}(\mathbf{p})$. A *direct evaluation* of the left-hand side of the above equations gives

$$\mathcal{G}(\mathbf{p}) = i\begin{pmatrix} 0 & 0 & 0 & -e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & -e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

It is to be immediately noted that $\mathcal{G}(\mathbf{p})$ is an odd function of $\mathbf{p}$

$$\mathcal{G}(\mathbf{p}) = -\mathcal{G}(-\mathbf{p}). \tag{24}$$

But since $\mathcal{G}(\mathbf{p})$ is independent of p and θ, it is more instructive to translate the above expression into

$$\mathcal{G}(\phi) = -\mathcal{G}(\pi + \phi). \tag{25}$$

This serves to define a preferred axis, z_e (see also Section 2.6 below).[4] Another hint for a preferred axis arises when one notes that the Elko spinorial structure does not enjoy covariance under usual local $U(1)$ transformation with phase $\exp(i\alpha(x))$. However, $U_E(1) = \exp(i\gamma^0\alpha(x))$ — and not $U_M(1) = \exp(i\gamma^5\alpha(x))$ as one would have thought [29, p. 72] — preserves various aspects of the Elko structure. Similar comments apply to the non-Abelian gauge transformations of the SM.

2.6. Elko and Dirac spinors: a comparison

For a comparison with the Dirac counterpart, one may define $g^\mu := (0, \mathbf{g})$ with

$$\mathbf{g} := -\left[1/\sin(\theta)\right]\partial\hat{\mathbf{p}}/\partial\phi = (\sin\phi, -\cos\phi, 0). \tag{26}$$

Note may be taken that g^μ is a unit spacelike four-vector, $g_\mu g^\mu = -1$. Furthermore, $g_\mu p^\mu = 0$. In terms of g^μ, $\mathcal{G}(\mathbf{p})$ may be written as

$$\mathcal{G}(\mathbf{p}) = \gamma^5(\gamma_1\sin\phi - \gamma_2\cos\phi) = \gamma^5\gamma_\mu g^\mu. \tag{27}$$

This gives Eqs. (21) and (22), the form

$$\sum_\alpha \xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p}) = m\left[\gamma^5\gamma_\mu g^\mu + \mathbb{I}\right], \tag{28}$$

$$\sum_\alpha \zeta_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p}) = m\left[\gamma^5\gamma_\mu g^\mu - \mathbb{I}\right]. \tag{29}$$

The appearance of g^μ on the right-hand side introduces a preferred axis.

The reader is reminded that so far no wave equation has been invoked. The charge conjugation and parity operators can be formally defined without reference to a wave equation. This can be seen from the fact that under parity $\kappa_r \leftrightarrow \kappa_\ell$, and thus the parity operator in the $r \oplus \ell$ representation space equals γ^0 (modulo a multiplicative phase factor). Dirac spinors then emerge as eigenspinors of the parity operator. From this perspective, when applied to eigenspinors of the parity operator, charge conjugation interchanges opposite parity eigenspinors (and it takes the form given in Eq. (5)). Once this view is accepted, one can start with an appropriate counterpart of the Elko at rest and following the same procedure as for Elko obtain the standard Dirac spinors, $u(\mathbf{p})$ and $v(\mathbf{p})$. The counterpart of the Elko spin sums then read

$$\sum_\alpha u_\sigma(\mathbf{p})\bar{u}_\sigma(\mathbf{p}) = m\left[m^{-1}\gamma_\mu p^\mu + \mathbb{I}\right], \tag{30}$$

$$\sum_\sigma v_\sigma(\mathbf{p})\bar{v}_\alpha(\mathbf{p}) = m\left[m^{-1}\gamma_\mu p^\mu - \mathbb{I}\right]. \tag{31}$$

The momentum-space Dirac equations now appear as identities derived from multiplying Eq. (30) from the right by $u_{\sigma'}(\mathbf{p})$, Eq. (31) by $v_{\sigma'}(\mathbf{p})$, and using $\bar{u}_\sigma(\mathbf{p})u_{\sigma'}(\mathbf{p}) = 2m\delta_{\sigma\sigma'}$ and $\bar{v}_\sigma(\mathbf{p})v_{\sigma'}(\mathbf{p}) = -2m\delta_{\sigma\sigma'}$. That these 'identities' are taken to lead to a wave equation, and eventually to derive the Lagrangian density, may have led to internal inconsistency unless the associated Green function was found to be proportional to $\langle\,|\mathcal{T}[\Psi(x')\overline{\Psi}(x)]|\,\rangle$, in the usual notation with $\Psi(x)$ as the Dirac quantum field. For the Dirac case this is precisely what happens and no internal inconsistency is introduced by following such a 'quick and dirty' route to arrive at the Lagrangian density.

To appreciate these remarks, a similar exercise may be undertaken for Elko. One finds that the resulting identities have no dynamical content.

2.7. Elko satisfy Klein–Gordon, not Dirac, equation

The next step in our discourse requires the observation that Elko do not satisfy the Dirac equation. To see this we apply the operator $\gamma^\mu p_\mu$ on Elko and find the following identities

$$\gamma^\mu p_\mu \xi_{\{-,+\}}(\mathbf{p}) = im\xi_{\{+,-\}}(\mathbf{p}), \tag{32}$$

$$\gamma^\mu p_\mu \xi_{\{+,-\}}(\mathbf{p}) = -im\xi_{\{-,+\}}(\mathbf{p}), \tag{33}$$

$$\gamma^\mu p_\mu \zeta_{\{-,+\}}(\mathbf{p}) = -im\zeta_{\{+,-\}}(\mathbf{p}), \tag{34}$$

$$\gamma^\mu p_\mu \zeta_{\{+,-\}}(\mathbf{p}) = im\zeta_{\{-,+\}}(\mathbf{p}). \tag{35}$$

Operating Eq. (32) from the left by $\gamma^\nu p_\nu$, and then using (33) on the resulting right-hand side, and repeating the same procedure for the remaining equations we get

$$\left(\gamma^\nu\gamma^\mu p_\nu p_\mu - m^2\right)\xi_{\{\mp,\pm\}}(\mathbf{p}) = 0, \tag{36}$$

$$\left(\gamma^\nu\gamma^\mu p_\nu p_\mu - m^2\right)\zeta_{\{\mp,\pm\}}(\mathbf{p}) = 0. \tag{37}$$

Now using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, yields the Klein–Gordon equation (in momentum space) for the $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ spinors. Aitchison

[4] The accompanying x_e and y_e axis help to define a preferred frame.

and Hey's concern [3, Appendix P] is thus overcome. The problem, as is now apparent, resides in the approach of constructing "simplest candidates for a kinematic spinor term" [30, p. 34]. The latter approach yields the "correct" results if Majorana spinors are treated as G-numbers, and the "wrong" result if they are treated as c-numbers. The systematic approach outlined here works in both contexts.

2.8. Two quantum fields with Elko as their expansion coefficients

We now examine the physical and mathematical content of two quantum fields with $\xi_\alpha(\mathbf{p})$ and $\zeta_\alpha(\mathbf{p})$ as their expansion coefficients

$$\Lambda(x) \stackrel{\text{def}}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\mathbf{p})}} \sum_\alpha \left[a_\alpha(\mathbf{p})\xi_\alpha(\mathbf{p})e^{-ip_\mu x^\mu} + b^\ddagger_\alpha(\mathbf{p})\zeta_\alpha(\mathbf{p})e^{+ip_\mu x^\mu}\right] \tag{38}$$

and

$$\lambda(x) \stackrel{\text{def}}{=} \Lambda(x)|_{b^\ddagger(\mathbf{p})\to a^\ddagger(\mathbf{p})}. \tag{39}$$

We assume that the annihilation and creation operators satisfy the fermionic anticommutation relations

$$\left\{a_\alpha(\mathbf{p}), a^\ddagger_{\alpha'}(\mathbf{p}')\right\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\alpha\alpha'}, \tag{40}$$

$$\left\{a_\alpha(\mathbf{p}), a_{\alpha'}(\mathbf{p}')\right\} = 0, \qquad \left\{a^\ddagger_\alpha(\mathbf{p}), a^\ddagger_{\alpha'}(\mathbf{p}')\right\} = 0. \tag{41}$$

Similar anticommutators are assumed for the $b_\alpha(\mathbf{p})$ and $b^\ddagger_\alpha(\mathbf{p})$. The adjoint field $\overset{\neg}{\Lambda}(x)$ is defined as

$$\overset{\neg}{\Lambda}(x) \stackrel{\text{def}}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\mathbf{p})}} \sum_\alpha \left[a^\ddagger_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p})e^{+ip_\mu x^\mu} + b_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p})e^{-ip_\mu x^\mu}\right]. \tag{42}$$

The results contained in Eqs. (32)–(35) assure us that it is the Klein–Gordon, and not the Dirac, operator that annihilates the fields $\Lambda(x)$ and $\lambda(x)$. The associated Lagrangian densities are

$$\mathcal{L}^\Lambda(x) = \partial^\mu\overset{\neg}{\Lambda}(x)\partial_\mu\Lambda(x) - m^2\overset{\neg}{\Lambda}(x)\Lambda(x), \tag{43}$$

$$\mathcal{L}^\lambda(x) = \mathcal{L}^\Lambda(x)|_{\Lambda\to\lambda}. \tag{44}$$

The mass dimensionality of these Elko fields is thus one, and not three half. Green functions and the consistency of these result with $\langle\ |\mathcal{T}[\Lambda(x')\overset{\neg}{\Lambda}(x)]|\ \rangle$ and $\langle\ |\mathcal{T}[\lambda(x')\overset{\neg}{\lambda}(x)]|\ \rangle$ shall be reported in an archival publication.

To study the locality structure of the fields $\Lambda(x)$ and $\lambda(x)$, we observe that field momenta are

$$\Pi(x) = \frac{\partial\mathcal{L}^\Lambda}{\partial\dot{\Lambda}} = \frac{\partial}{\partial t}\overset{\neg}{\Lambda}(x), \tag{45}$$

and similarly $\pi(x) = \frac{\partial}{\partial t}\overset{\neg}{\lambda}(x)$. The calculational details for the two fields now differ significantly. We begin with the evaluation of the equal time anticommutator for $\Lambda(x)$ and its conjugate momentum

$$\left\{\Lambda(\mathbf{x},t), \Pi(\mathbf{x}',t)\right\} = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2m}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \times \underbrace{\sum_\alpha\left[\xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(-\mathbf{p})\overset{\neg}{\zeta}_\alpha(-\mathbf{p})\right]}_{=2m[\mathbb{I}+\mathcal{G}(\mathbf{p})]}.$$

The term containing $\mathcal{G}(\mathbf{p})$ vanishes only when $\mathbf{x}-\mathbf{x}'$ lies along the z_e axis (see Eq. (24), and discussion of this integral in Refs. [1,2])

$$\mathbf{x}-\mathbf{x}' \text{ along } z_e: \quad \left\{\Lambda(\mathbf{x},t), \Pi(\mathbf{x}',t)\right\} = i\delta^3(\mathbf{x}-\mathbf{x}')\mathbb{I}. \tag{46}$$

The anticommutators for the particle/antiparticle annihilation and creation operators suffice to yield the remaining locality conditions,

$$\left\{\Lambda(\mathbf{x},t), \Lambda(\mathbf{x}',t)\right\} = \mathbb{O}, \qquad \left\{\Pi(\mathbf{x},t), \Pi(\mathbf{x}',t)\right\} = \mathbb{O}. \tag{47}$$

The set of anticommutators contained in Eqs. (46) and (47) establish that $\Lambda(x)$ becomes local along the z_e axis. For this reason we call z_e as the dark axis of locality.

For the equal time anticommutator of the $\lambda(x)$ field with its conjugate momentum, we find

$$\left\{\lambda(\mathbf{x},t), \pi(\mathbf{x}',t)\right\} = i\int\frac{d^3p}{(2\pi)^3}\frac{1}{2m} \times \sum_\alpha\left[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\left(\xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(-\mathbf{p})\overset{\neg}{\zeta}_\alpha(-\mathbf{p})\right)\right].$$

Which, using similar arguments as before, yields

$$\mathbf{x}-\mathbf{x}' \text{ along } z_e: \quad \left\{\lambda(\mathbf{x},t), \pi(x',t)\right\} = i\delta^3(\mathbf{x}-\mathbf{x}')\mathbb{I}. \tag{48}$$

The difference arises in the evaluation of the remaining anticommutators. The equal time λ–λ anticommutator reduces to

$$\left\{\lambda(\mathbf{x},t), \lambda(\mathbf{x}',t)\right\} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\mathbf{p})}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \times \underbrace{\sum_\alpha\left[\xi_\alpha(\mathbf{p})\zeta^T_\alpha(\mathbf{p}) + \zeta_\alpha(-\mathbf{p})\xi^T_\alpha(-\mathbf{p})\right]}_{=:\Omega(\mathbf{p})}. \tag{49}$$

Now using explicit expressions for $\xi_\alpha(\mathbf{p})$ and $\zeta_\alpha(\mathbf{p})$ we find that $\Omega(\mathbf{p})$ identically vanishes. Eq. (49) then implies

$$\left\{\lambda(\mathbf{x},t), \lambda(\mathbf{x}',t)\right\} = \mathbb{O}. \tag{50}$$

And, finally the equal time π–π anticommutator simplifies to

$$\left\{\pi(\mathbf{x},t), \pi(\mathbf{x}',t)\right\} = \int\frac{d^3p}{(2\pi)^3}\frac{E(\mathbf{p})}{2m}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \times \underbrace{\sum_\alpha\left[\left(\overset{\neg}{\xi}_\alpha(\mathbf{p})\right)^T\overset{\neg}{\zeta}_\alpha(\mathbf{p}) + \left(\overset{\neg}{\zeta}_\alpha(-\mathbf{p})\right)^T\overset{\neg}{\xi}_\alpha(-\mathbf{p})\right]}_{=\mathbb{O},\ \text{by a direct evaluation}},$$

yielding

$$\left\{\pi(\mathbf{x},t), \pi(\mathbf{x}',t)\right\} = \mathbb{O}. \tag{51}$$

Again, $\lambda(x)$ becomes local along z_e. This further justifies the term 'dark axis of locality' for the z_e axis.

The dimension four interactions of the $\Lambda(x)$ and $\lambda(x)$ with the standard model fields are restricted to those with the SM Higgs doublet $\phi(x)$. These are

$$\mathcal{L}^{\text{int}}(x) = \phi^\dagger(x)\phi(x)\sum_{\psi,\Psi}a_{\psi\Psi}\overset{\neg}{\psi}(x)\Psi(x), \tag{52}$$

where $a_{\psi\Psi}$ are unknown coupling constants and symbols ψ and Ψ stand for either Λ or λ. By virtue of their mass dimensionality the new Elko fields are endowed with dimension four quartic self interactions contained in

$$\mathcal{L}^{\text{self}} = \sum_{\psi,\Psi} b_{\psi\Psi}\left[\overset{\neg}{\psi}(x)\Psi(x)\right]^2, \tag{53}$$

where $b_{\psi\Psi}$ are unknown coupling constants.

Remarks following Eq. (25) suggest that the Elko fields need not be self referentially dark. However, the same remarks imply that quantum fields based on Elko may not participate in interactions with the standard model gauge fields. This also allows the Elko-based dark matter to evade the constraints on preferred-frame effects discussed in literature (see, e.g., Ref. [31]).

3. Concluding remarks

This Letter is a natural and non-trivial continuation of the 2005 work of Ahluwalia and Grumiller on Elko. Here we reported that Elko breaks Lorentz symmetry in a rather subtle and unexpected way by containing a 'hidden' preferred direction. Along this preferred direction, a quantum field based on Elko enjoys locality. In the form reported here, Elko offers mass dimension one fermionic dark matter with a quartic self-interaction and a preferred axis of locality. The locality result crucially depends on a judicious choice of phases.

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Self-interacting Elko dark matter with an axis of locality

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Here we report that Elko (for ***E****igenspinoren des* **L***adungs***k***onjugations***o***perators*) breaks Lorentz symmetry in a rather subtle and unexpected way by containing a ''hidden'' preferred direction. Along this preferred direction, a quantum field based on Elko enjoys locality. In the form reported here, Elko offers a mass dimension one fermionic dark matter with a quartic self-interaction and a preferred axis of locality. The locality result crucially depends on a judicious choice of phases.

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I. INTRODUCTION

The particle nature of dark matter is still unsettled. What we do know is that it is expected to be endowed with a self-interaction [1–4]. The indicated self-interaction would ordinarily suggest that dark matter must be some sort of scalar field. However, as shown in [5,6], the Elko (for ***E****igenspinoren des* **L***adungs***k***onjugations***o***perators*, the reason for this definition will become clear in Sec. I B) quantum field is endowed with mass-dimension one, a property that allows for unsuppressed Elko self-interaction. Further consequences of the mass dimensionality of Elko are that its possible interactions with the mass-dimension 3/2 Dirac and Majorana fields are suppressed by one order of unification scale and that it cannot enter the standard model (SM) doublets. This, along with the fact that Elko does not carry the standard U(1) gauge invariance, renders Elko a natural dark matter candidate [5,6].

Here we report that Elko breaks Lorentz symmetry in a rather subtle and unexpected way by containing a ''hidden'' preferred direction. All inertial frames that move with a constant velocity along this direction are physically equivalent. Along this direction, a quantum field based on Elko enjoys locality.

Our discourse begins with a review of the SM matter fields in Sec. I A. In Sec. I B we recapitulate the known problems with the interpretation of Majorana spinors as commuting numbers, and argue that these problems evaporate under a more careful examination [5,6]. The pace is deliberately slow. The discussion is designed to provide the right setting for the taken departure. Sections II and III form the core of this communication. The discussion on the Elko dual presented in Sec. II B is a significant addition to the previous work on Elko [5,6]. The dramatically changed locality structure arises from certain phases and identification introduced in the Elko spinors at rest [see Eqs. (16a)–(16d)]. Section II C reminds the reader that Elko satisfies the Klein-Gordon, but not the Dirac, equation. The Elko spin sums are given in Sec. II D. These spin sums are needed for examining the locality structure of the Elko quantum fields and had to be reevaluated due to the mentioned changes in the Elko rest spinors [7]. These carry the seeds of the mentioned preferred direction. Section III formally introduces the Elko quantum fields. Section III A makes an argument to identify Elko with self-interacting dark matter that is endowed with an axis of locality. In the form reported here, Elko offers a mass-dimension one fermionic dark matter with self-interaction and a preferred axis of locality. The locality result crucially depends on a judicious choice of phases. The paper ends with summarizing remarks and questions in Sec. IV. Appendix A provides supplementary information.

A. The matter field underlying the SM

The matter field underlying the SM is a four-component spinor field [8] with historical origin in Dirac's celebrated 1928 paper [9]

$$\Psi(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E(\mathbf{p})}} \times [\underbrace{u(x;\mathbf{p},\sigma)}_{=u(\mathbf{p},\sigma)e^{-ip^\mu x_\mu}} a(\mathbf{p},\sigma) + \underbrace{v(x;\mathbf{p},\sigma)}_{=v(\mathbf{p},\sigma)e^{+ip^\mu x_\mu}} b^\dagger(\mathbf{p},\sigma)], \tag{1}$$

where σ takes the values $\pm 1/2$. The zero-momentum coefficient functions may be symbolically written as

$$u(0,1/2) = \begin{bmatrix}\uparrow\\ \uparrow\end{bmatrix}, \qquad u(0,-1/2) = \begin{bmatrix}\downarrow\\ \downarrow\end{bmatrix}, \tag{2}$$

$$v(0,1/2) = \begin{bmatrix}\downarrow\\ -\downarrow\end{bmatrix}, \qquad v(0,-1/2) = \begin{bmatrix}-\uparrow\\ \uparrow\end{bmatrix}, \tag{3}$$

where

$$\uparrow \equiv \sqrt{m}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \downarrow \equiv \sqrt{m}\begin{pmatrix}0\\1\end{pmatrix} \tag{4}$$

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in the "polarization basis." In the helicity basis, these are eigenspinors of the helicity operator with a specific choice of phases. These phases are determined, e.g., by the locality condition [10].

Without any reference to the Dirac equation (see Ref. [8] for a detailed argument), the coefficient functions are determined from the condition that, under the homogeneous Lorentz transformations, the field components superimpose with other field components via spacetime-independent elements (of 4×4 matrices). These matrices must furnish a finite dimensional representation of the homogeneous Lorentz group.

The coefficient functions for arbitrary momentum are obtained by the action of the boost

$$u(\mathbf{p}, \sigma) = \kappa u(0, \sigma), \tag{5}$$

where $\kappa \equiv \kappa_r \oplus \kappa_\ell$. The explicit expressions for κ_r and κ_ℓ are given below.

The only nontrivial freedom that $\Psi(x)$ still carries is the specialization to the case where $b^\dagger(\mathbf{p}, \sigma)$ is identified with $a^\dagger(\mathbf{p}, \sigma)$. Otherwise, the Poincaré spacetime symmetries along with the symmetries of charge-conjugation, parity and time-reversal, and the demand of locality uniquely determine the field $\Psi(x)$. Seen in this light, the field coefficients $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ are eigenspinors of the $\gamma^\mu p_\mu$ operator with eigenvalues $+m$ and $-m$, respectively.

The annihilation of the field $\Psi(x)$ by the Dirac operator $(i\gamma^\mu \partial_\mu - m\mathbb{I})$ follows as a result of this structure. The Dirac equation is not assumed. Rather, it emerges as a direct consequence of the merger of quantum mechanics and Poincaré spacetime symmetries for spin $1/2$. The apparent simplicity of the Dirac field can be somewhat misleading to the uninitiated. For instance, a change in sign in the right-hand side of the expression for $v(0, -1/2)$ in Eq. (3) yields a quantum field that is nonlocal when $b^\dagger(\mathbf{p}, \sigma)$ is identified with $a^\dagger(\mathbf{p}, \sigma)$. Even though the mentioned change in phase does not destroy the locality in the original field, it does violate spacetime symmetries in a hidden way. A systematic study of such subtle loss of symmetries and locality remains largely unexplored.

For historical reasons, the field $\Psi(x)$ is known as the Dirac field, while the identification of $b^\dagger(\mathbf{p}, \sigma)$ with $a^\dagger(\mathbf{p}, \sigma)$ yields what has come to be known as the Majorana field [9,11]. The coefficient functions $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ are the usual Dirac spinors. They can be interpreted as being a direct sum of the right-handed and left-handed Weyl spinors with specific helicities and phases.

B. Majorana spinors: A critique

History clearly demarcates the introduction of the Majorana field. It was introduced in 1937 by Ettore Majorana [11]. As regards Majorana spinors, we (i.e., the authors) do not know of their historical birth.

While in the operator formalism of quantum field theory Dirac spinors are treated as commuting numbers, it is curious that Majorana spinors are treated as Grassmann variables. This is deemed necessary, due to what are considered otherwise unavoidable problems. (Consider for instance Aitchison and Hey's attempt to construct a Hamiltonian density [12].) What further adds to the problem is that, taken by itself, a Majorana spinor is nothing but a Weyl spinor in the four-component form. As shown by Ahluwalia and Grumiller [5,6], both of these problems can be circumvented. A hint toward a solution for the first problem may be found by noting that, unlike Dirac spinors, the Majorana spinors are not eigenspinors of the Dirac operator. Instead, they are eigenspinors of the square of the Dirac operator. This suggests that the problem lies not with the Majorana spinors but instead with the Lagrangian density assumed by Aitchison and Hey [12]. The latter of the two mentioned problems also has a similar solution. The usual set of Majorana spinors consists of two spinors, both of which have eigenvalue one under the operation of the charge conjugation operator. This is the self-conjugate set. However, as pointed out in Refs. [5,6], there also exists the anti self-conjugate set. Once these are added, the complete set of four spinors—the Elko (for ***E****igenspinoren des* **L***adungs***k***onjugations* ***o****perators*)—span the four-dimensional representation space of spin $1/2$ and come to par with the Dirac spinors.

Let us briefly review the canonical wisdom. In doing so we shall explicitly show the cost at which the above changes are implemented. Whether or not this ought to be a cost we should be willing to pay is ultimately a matter for experiment to decide. At the very least, we shall know what it is that we would reject if we were to choose to confine ourselves to the canonical wisdom.

According to the received wisdom, the Majorana spinors airse as follows. If ϕ_ℓ is a massive Weyl spinor of left-handed nature, then $\sigma_2 \phi_\ell^*$ transforms as a right-handed Weyl spinor. For this reason ([13], p. 20), we can construct a special type of four-component spinor called a Majorana spinor:

$$\psi_M = \begin{pmatrix} -\sigma_2 \phi_\ell^* \\ \phi_\ell \end{pmatrix}. \tag{6}$$

It is self-conjugate under charge conjugation. For ϕ_ℓ there are two choices: a positive helicity and a negative helicity. As such, we have two rather than four four-component spinors. Thus the folklore: the Majorana spinor is a Weyl spinor in four-component form [13]. It is self-evident and remains unquestioned in our discourse.

An immediate sign of trouble appears if one naïvely introduces a Lagrangian density $\mathcal{L}_M = \bar{\psi}_M (i\gamma^\mu \partial_\mu - m)\psi_M$. The usual route at this stage is to treat the components of the Weyl spinors as Grassmann numbers; otherwise, one encounters the often-quoted problems ([12], App. P). The Ahluwalia-Grumiller work [5,6] strongly indicates that this approach may be hiding certain fundamental properties of Majorana spinors. Or, to put it more precisely, having taken the Grassmann route, we may have overlooked a rich and fertile ground where Majorana spinors are treated as commuting number spinors. To unearth these aspects, we shall treat the massive Weyl spinors as two-component eigenspinors of the helicity operator ([14], p. 111). The fermionic statistics are implemented through the canonical field operator formalism [8,15] and not by treating them as Grassmann fields [16]. The Elko formalism was born in this spirit and attended to a widespread, but rarely spoken, discontent with abandoning Majorana spinors as commuting numbers.

A straightforward calculation now shows that, (i) under the Dirac dual, the norm $\bar{\psi}_M \psi_M$ identically vanishes (so, no Dirac mass term); and (ii) in the momentum space, ψ_M is not an eigenspinor of the $\gamma_\mu p^\mu$ operator $\gamma_\mu p^\mu \psi_M \neq \pm m \psi_M$ (and so Majorana spinors do not satisfy the Dirac equation ([12], App. P). This already suggests that constructing a mass dimension $3/2$ fermionic field in terms of Majorana spinors may not be possible [18]. The lesson to be learned is this: It is *not* sufficient that one consider the "simplest candidates for a kinematic spinor term" in the construction of a field equation, as found in almost [19] every text book on quantum field theory [21]. Rather, one must ensure that the associated Green's function be proportional to the vacuum expectation value of the time-ordered product of certain field operators. This lesson, we think, has a much larger significance in that Lagrangian densities must be derived and not assumed. Neglecting this may induce all manner of pathologies. How this task is to be accomplished—at least for spin $1/2$—is one of the wider contributions of this communication.

The assertion about reduction in the degrees of freedom for Majorana spinors also faces trouble if one notes that the relevant charge conjugation operator has not one, but two, real eigenvalues: $+1$ (giving the usual self-conjugate Majorana spinors) and -1. There is no physical or mathematical reason to abandon, or "project out," the latter. The sense in which the folklore still survives is that, by an appropriate similarity transformation, half of these (i.e., those corresponding to the positive eigenvalue) can be mapped to real four-component spinors, while those corresponding to the negative eigenvalue can be transformed into purely imaginary four-component spinors.

II. ELKO: DEPARTURE FROM GRASSMANN INTERPRETATION OF MAJORANA SPINORS

The interpretation of the Majorana spinors in terms of Grassmann variables is elegant. It is mathematically sound and has found widespread applications in modern quantum field theory. Yet it breaks with the tradition of field operator formalism which would have required these spinors to be commuting number coefficient functions in a field. In their work [5,6], Ahluwalia and Grumiller formulated a treatment of Majorana spinors in the operator formalism. Towards this end, they included two additional spinors to the canonical Majorana spinors, thus forming a complete set of dual helicity eigenspinors of the charge conjugation operator for spin $1/2$. In order to avoid confusion with the incomplete set, the Majorana spinors, they introduced the name Elko, which, as already mentioned, was taken from the German *Eigenspinoren des Ladungskonjugationsoperators*.

The quantum field expanded with Elko spinors is not a quantum field in the sense of Weinberg [8]. Specifically, the uniqueness of the Dirac field, modulo its specialization to the Majorana field, implies that the program we embark upon necessarily violates Lorentz symmetry. This feature, which had remained hidden in our previous discourse, we now unearth. In our opinion, this has the potential to open up an entirely new perspective on dark matter—the decision being in the hands of experiments. To a pure theoretician, the interest might be in its mathematical structure.

In this communication we confine our primary attention to spin $1/2$, but we construct Elko in such a way that the procedure immediately generalizes to all spins. This is facilitated by the use of Wigner's time-reversal operator $\Theta = -i\sigma_2$, rather than the Pauli matrix σ_2 that appears in Ramond's primer in the context of Majorana spinors. We shall use the phrase Elko for spinors as well as for the quantum fields constructed from them. The context shall be assumed to remove any ambiguity.

A. Construction of Elko

To construct Elko it is first necessary to introduce the charge conjugation operator. This we do as follows. Under parity, P, $\mathbf{x} \to -\mathbf{x}$; hence, the rapidity parameter $\boldsymbol{\varphi} = \varphi\hat{\mathbf{p}}$ changes sign. Thus, to implement this transformation on the boost operator, we require a matrix of the form

$$S(P) = \exp[i\vartheta]\underbrace{\begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}}_{\gamma^0}\mathcal{R}, \qquad \vartheta \in \mathbb{R}, \tag{7}$$

with $\mathbf{p} \equiv p(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, and $\mathcal{R} = \{\theta \to \pi - \theta, \phi \to \pi + \phi, p \to p\}$. If care is taken that the eigenvalues of the helicity operator change sign under P, the arguments given in Ref. [6] fix the phase $\exp[i\vartheta]$ to be i. The operator $S(P)$ now has four doubly degenerate eigenspinors, carrying opposite eigenvalues of $S(P)$—call these u and v sectors. The operator

$$\mathcal{C} = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix}K, \tag{8}$$

where K is the complex conjugation operator, formally interchanges the opposite parity sectors: $u \overset{\mathcal{C}}{\leftrightarrow} v$. It is apparent that $\mathcal{C}$ is the standard charge conjugation operator of Dirac. In the context of Eq. (8), Wigner's time-reversal operator Θ is defined as $\Theta\mathbf{J}\Theta^{-1} = -\mathbf{J}^*$, where $\mathbf{J}$ are a set of rotation generators for the representation space under consideration. For spin 1/2, $\Theta[\boldsymbol{\sigma}/2]\Theta^{-1} = -[\boldsymbol{\sigma}/2]^*$. We use the realization

$$\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

To construct Elko, let $\phi_\ell(\mathbf{p})$ be a left-handed Weyl spinor of spin 1/2. Under a Lorentz boost, it transforms as $\phi_\ell(\mathbf{p}) = \kappa_\ell \phi_\ell(\boldsymbol{\epsilon})$, with

$$\kappa_\ell = \exp\left(-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right) = \sqrt{\frac{E+m}{2m}}\left(\mathbb{I} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right). \tag{9}$$

The $\boldsymbol{\epsilon}$ is defined as $\mathbf{p}|_{p\to 0}$, and not as $\mathbf{p}|_{p=0}$. In the usual notation, the boost parameter $\boldsymbol{\varphi}$ is defined as

$$\cosh\varphi = \frac{E}{m} = \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \sinh\varphi = \frac{p}{m} = \gamma\beta, \qquad \hat{\boldsymbol{\varphi}} = \hat{\mathbf{p}}. \tag{10}$$

By $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ we denote the Pauli matrices. The symbol $\mathbb{I}$ represents an identity matrix, while in what follows $\mathbb{O}$ shall be used for a null matrix (their dimensionality shall be apparent from the context). For $\phi_\ell(\mathbf{p})$, we have two possibilities:

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\phi_\ell^\pm(\mathbf{p}) = \pm\phi_\ell^\pm(\mathbf{p}).$$

Following Ref. [6] we now note that, under a Lorentz boost, $\vartheta\Theta\phi_\ell^*(\mathbf{p})$ transforms as a right-handed Weyl spinor, $[\vartheta\Theta\phi_\ell^*(\mathbf{p})] = \kappa_r[\vartheta\Theta\phi_\ell^*(\boldsymbol{\epsilon})]$, with

$$\kappa_r = \exp\left(+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right) = \sqrt{\frac{E+m}{2m}}\left(\mathbb{I} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right), \tag{11}$$

where ϑ is an unspecified phase to be determined below. The helicity of $\vartheta\Theta\phi_\ell^*(\mathbf{p})$ is *opposite* to that of $\phi_\ell(\mathbf{p})$,

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}[\vartheta\Theta(\phi_\ell^\pm(\mathbf{p}))^*] = \mp[\vartheta\Theta(\phi_\ell^\pm(\mathbf{p}))^*]. \tag{12}$$

The argument that led to *two* Majorana spinors, now instead takes us to their cousins, the *four* four-component spinors with the general form

$$\chi(\mathbf{p}) = \begin{pmatrix} \vartheta\Theta\phi_\ell^*(\mathbf{p}) \\ \phi_\ell(\mathbf{p}) \end{pmatrix}. \tag{13}$$

The $\chi(\mathbf{p})$ become eigenspinors of the charge conjugation operator, Elko, with real eigenvalues if the phase ϑ is restricted to $\pm i$:

$$\mathcal{C}\chi(\mathbf{p})|_{\vartheta=\pm i} = \pm\chi(\mathbf{p})|_{\vartheta=\pm i}. \tag{14}$$

One can motivate the well-known Dirac spinors in a parallel manner; as eigenspinors of the parity operator $S(P)$. In that case, the right- and left-transforming components are necessarily endowed with the same helicity. For Elko, the right- and left-transforming components carry opposite helicity. So, whereas Dirac spinors may exist as eigenspinors of the helicity operator, the Elko cannot. This eventually is reflected in many of the results that we arrive at.

To give Elko a concrete form, we adopt the global phases so that, "at rest," the left-handed Weyl spinors take the form [22]

$$\phi_\ell^+(\boldsymbol{\epsilon}) = \sqrt{m}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}, \tag{15a}$$

$$\phi_\ell^-(\boldsymbol{\epsilon}) = \sqrt{m}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix}. \tag{15b}$$

Eqs. (15a) and (15b), along with Eq. (13) and the demand of locality allow us to explicitly write the self-conjugate spinors $(\vartheta = +i)$ and anti–self-conjugate spinors $(\vartheta = -i)$ at rest:

$$\xi_{\{-,+\}}(\boldsymbol{\epsilon}) \equiv +\chi(\boldsymbol{\epsilon})|_{\phi_\ell(\boldsymbol{\epsilon})\to\phi_\ell^+(\boldsymbol{\epsilon}),\vartheta=+i} \tag{16a}$$

$$\xi_{\{+,-\}}(\boldsymbol{\epsilon}) \equiv +\chi(\boldsymbol{\epsilon})|_{\phi_\ell(\boldsymbol{\epsilon})\to\phi_\ell^-(\boldsymbol{\epsilon}),\vartheta=+i} \tag{16b}$$

$$\zeta_{\{-,+\}}(\boldsymbol{\epsilon}) \equiv +\chi(\boldsymbol{\epsilon})|_{\phi_\ell(\boldsymbol{\epsilon})\to\phi_\ell^-(\boldsymbol{\epsilon}),\vartheta=-i} \tag{16c}$$

$$\zeta_{\{+,-\}}(\boldsymbol{\epsilon}) \equiv -\chi(\boldsymbol{\epsilon})|_{\phi_\ell(\boldsymbol{\epsilon})\to\phi_\ell^+(\boldsymbol{\epsilon}),\vartheta=-i}. \tag{16d}$$

For comparison with Eqs. (2)–(4), the above in polarization basis may be written as

$$\xi_{\{-,+\}}(\boldsymbol{\epsilon}) = \begin{bmatrix} i\Downarrow \\ \Uparrow \end{bmatrix}, \qquad \xi_{\{+,-\}}(\boldsymbol{\epsilon}) = \begin{bmatrix} -i\Uparrow \\ \Downarrow \end{bmatrix}, \tag{17}$$

$$\zeta_{\{-,+\}}(\boldsymbol{\epsilon}) = \begin{bmatrix} i\Uparrow \\ \Downarrow \end{bmatrix}, \qquad \zeta_{\{+,-\}}(\boldsymbol{\epsilon}) = -\begin{bmatrix} -i\Downarrow \\ \Uparrow \end{bmatrix}. \tag{18}$$

The $\Uparrow$ and $\Downarrow$ differ from $\uparrow$ and $\downarrow$ of Eq. (4) by the phases, $e^{\pm i\phi/2}$, which even in the polarization basis prove to be essential if locality is to be preserved. In the context of Weinberg's work on the uniqueness of the Dirac field (modulo the special case of the Majorana field in the sense of Majorana's original 1937 paper [11]), a comparison with Eqs. (2) and (3) already tells us that a quantum field that fully respects Lorentz symmetries cannot be built in terms of ξ and ζ Elko spinors. The task then is to unearth this violation, and see how strong, or how weak, the said violation is.

The $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ for an arbitrary momentum are now readily obtained:

$$\xi(\mathbf{p}) = \kappa\xi(\boldsymbol{\epsilon}), \qquad \zeta(\mathbf{p}) = \kappa\zeta(\boldsymbol{\epsilon}), \qquad \kappa \equiv \kappa_r \oplus \kappa_\ell. \tag{19}$$

B. A systematic construction of Elko dual, orthonormality, and completeness

The norm of Elko under the Dirac dual $\bar{\chi}(\mathbf{p}) \equiv [\chi(\mathbf{p})]^\dagger\gamma^0$ identically vanishes. However, it is more appropriate to seek a "metric" η such that the product $[\chi_i(\mathbf{p})]^\dagger\eta\chi_j(\mathbf{p})$—with $\chi_i(p)$ as any one of the four Elko spinors—remains invariant under an arbitrary Lorentz transformation. This requirement can be readily shown to translate into the following constraints on η:

$$[J_i, \eta] = 0, \qquad \{K_i, \eta\} = 0. \tag{20}$$

Since the only property of the generators of rotations and boosts that enters the derivation of the above constraints is that $\mathbf{J}^\dagger = \mathbf{J}$ and $\mathbf{K}^\dagger = -\mathbf{K}$, the result applies to all *finite* dimensional representations of the Lorentz group. It need not be restricted to Elko alone. Seen in this light, there is no nontrivial solution for η for either the right-handed or the left-handed Weyl spinors. For $r \oplus \ell$ representation space, the most general solution is found to carry the form

$$\eta = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}. \tag{21}$$

It is now convenient to introduce the notation $\chi_1(\mathbf{p}) \equiv \xi_{\{-,+\}}(\mathbf{p})$, $\chi_2(\mathbf{p}) \equiv \xi_{\{+,-\}}(\mathbf{p})$, $\chi_3(\mathbf{p}) \equiv \zeta_{\{-,+\}}(\mathbf{p})$, and $\chi_4(\mathbf{p}) \equiv \zeta_{\{+,-\}}(\mathbf{p})$. Then 16 values of $[\chi_i(\mathbf{p})]^\dagger \eta \chi_j(\mathbf{p})$ as i and j vary from 1 to 4 are given in Table I.

To allow for the possibility of parity covariance, we set $b = a$. (This treats r and ℓ Weyl spaces on the same footing.) To make the invariant norms real, we give a and b the common value of $\pm i$, resulting in $\eta = \pm i\gamma^0$. In what follows, the choice of the signs shall be dictated by the convenience of bookkeeping.

Guided by these results, we now introduce the *Elko dual*

$$\neg{\chi}_{\{\mp,\pm\}}(\mathbf{p}) \equiv \mp i[\chi_{\{\pm,\mp\}}(\mathbf{p})]^\dagger \gamma^0. \tag{22}$$

Under the new dual, the orthonormality relations read

$$\neg{\xi}_\alpha(\mathbf{p})\xi_{\alpha'}(\mathbf{p}) = +2m\delta_{\alpha\alpha'}, \tag{23a}$$

$$\neg{\zeta}_\alpha(\mathbf{p})\zeta_{\alpha'}(\mathbf{p}) = -2m\delta_{\alpha\alpha'}, \tag{23b}$$

along with $\neg{\xi}_\alpha(\mathbf{p})\zeta_{\alpha'}(\mathbf{p}) = 0$ and $\neg{\zeta}_\alpha(\mathbf{p})\xi_{\alpha'}(\mathbf{p}) = 0$. The dual helicity index α ranges over the two possibilities: $\{+,-\}$ and $\{-,+\}$, and $-\{\pm,\mp\} \equiv \{\mp,\pm\}$. The completeness relation

$$\frac{1}{2m}\sum_\alpha [\xi_\alpha(\mathbf{p})\neg{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(\mathbf{p})\neg{\zeta}_\alpha(\mathbf{p})] = \mathbb{I} \tag{24}$$

establishes that we need to use *both* the self-conjugate as well as the anti–self-conjugate spinors to fully capture the relevant degrees of freedom.

TABLE I. The values of $[\chi_i(\mathbf{p})]^\dagger \eta \chi_j(\mathbf{p})$ evaluated using η. The i runs from 1 to 4 along the rows and j does the same across the columns.

0	$-im(a+b)$	$-im(a-b)$	0
$im(a+b)$	0	0	$-im(a-b)$
$-im(a-b)$	0	0	$im(a+b)$
0	$-im(a-b)$	$-im(a+b)$	0

C. Elko satisfies the Klein-Gordon, not Dirac, equation

Because we are going to encounter several unexpected results, we pause to examine the behavior of $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ spinors under the action of the operator $\gamma^\mu p_\mu$. This brute force exercise serves the pedagogic purpose of countering some prejudices some readers may inevitably carry from their prior studies. Additionally, in the context of Aitchison and Hey's concern that one encounters a problem in constructing a Lagrangian density for Majorana spinors if they are not treated as Grassmann variables ([12], App. P), we provide the origin of that concern and offer a solution.

We already have explicit expressions for $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ spinors. On these we act $\gamma^\mu p_\mu$ and find the following identities:

$$\gamma^\mu p_\mu \xi_{\{-,+\}}(\mathbf{p}) = +im\xi_{\{+,-\}}(\mathbf{p}) \Leftrightarrow \gamma^\mu p_\mu \chi_1(\mathbf{p}) = +im\chi_2(\mathbf{p}) \tag{25a}$$

$$\gamma^\mu p_\mu \xi_{\{+,-\}}(\mathbf{p}) = -im\xi_{\{-,+\}}(\mathbf{p}) \Leftrightarrow \gamma^\mu p_\mu \chi_2(\mathbf{p}) = -im\chi_1(\mathbf{p}) \tag{25b}$$

$$\gamma^\mu p_\mu \zeta_{\{-,+\}}(\mathbf{p}) = -im\zeta_{\{+,-\}}(\mathbf{p}) \Leftrightarrow \gamma^\mu p_\mu \chi_3(\mathbf{p}) = -im\chi_4(\mathbf{p}) \tag{25c}$$

$$\gamma^\mu p_\mu \zeta_{\{+,-\}}(\mathbf{p}) = +im\zeta_{\{-,+\}}(\mathbf{p}) \Leftrightarrow \gamma^\mu p_\mu \chi_4(\mathbf{p}) = +im\chi_3(\mathbf{p}). \tag{25d}$$

Applying $\gamma^\nu p_\nu$ to Eq. (25a) from the left and then using (25b) on the resulting right-hand side, and repeating the same procedure for the remaining equations, we get

$$\begin{aligned} (\gamma^\nu\gamma^\mu p_\nu p_\mu - m^2)\xi_{\{\mp,\pm\}}(\mathbf{p}) &= 0, \\ (\gamma^\nu\gamma^\mu p_\nu p_\mu - m^2)\zeta_{\{\mp,\pm\}}(\mathbf{p}) &= 0. \end{aligned} \tag{26}$$

Now using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ yields the Klein-Gordon equation (in momentum space) for the $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ spinors. Aitchison and Hey's concern is thus overcome. The problem resides in the approach of constructing the "simplest candidates for a kinematic spinor term."

D. Elko spin sums: A preferred axis

We now look at the spin sums in Eq. (24) separately. These evaluate to

$$\sum_\alpha \xi_\alpha(\mathbf{p})\neg{\xi}_\alpha(\mathbf{p}) = m[\mathcal{G}(\mathbf{p}) + \mathbb{I}], \tag{27a}$$

$$\sum_\alpha \zeta_\alpha(\mathbf{p})\neg{\zeta}_\alpha(\mathbf{p}) = m[\mathcal{G}(\mathbf{p}) - \mathbb{I}], \tag{27b}$$

which together *define* $\mathcal{G}(\mathbf{p})$. A *direct evaluation* of the left-hand side of the above equations gives

$$\mathcal{G}(\mathbf{p}) = i\begin{pmatrix} 0 & 0 & 0 & -e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & -e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

For later reference, we note that $\mathcal{G}(\mathbf{p})$ is an odd function of $\mathbf{p}$:

$$\mathcal{G}(\mathbf{p}) = -\mathcal{G}(-\mathbf{p}). \tag{29}$$

But since $\mathcal{G}(\mathbf{p})$ is independent of p and θ, it is more instructive to translate the above expression into

$$\mathcal{G}(\phi) = -\mathcal{G}(\pi + \phi). \tag{30}$$

This serves to define a preferred axis, z_e [23]. Another hint for a preferred axis arises when one notes that the spinor structure of Elko does not enjoy covariance under usual local $U(1)$ transformation with phase $\exp(i\alpha(x))$. However, $U_E(1) = \exp(i\gamma^0\alpha(x))$—and not $U_M(1) = \exp(i\gamma^5\alpha(x))$ as one would have thought ([24], p. 72)—preserves various aspects of the Elko structure. Similar comments apply to the non-Abelian gauge transformations of the SM.

For a comparison with the Dirac counterpart (see App. A 1), we define $g^\mu \equiv (0, \mathbf{g})$ with $\mathbf{g} = -[1/\sin(\theta)]\partial\hat{\mathbf{p}}/\partial\phi = (\sin\phi, -\cos\phi, 0)$. Note may be taken that g^μ is a unit spacelike four-vector, $g_\mu g^\mu = -1$. Furthermore, $g_\mu p^\mu = 0$. In terms of g^μ, $\mathcal{G}(\mathbf{p})$ may be written as

$$\mathcal{G}(\mathbf{p}) = \gamma^5(\gamma_1 \sin\phi - \gamma_2 \cos\phi) = \gamma^5\gamma_\mu g^\mu. \tag{31}$$

This gives Eqs. (27a) and (27b) the form

$$\sum_\alpha \xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p}) = m[\gamma^5\gamma_\mu g^\mu + \mathbb{I}], \tag{32a}$$

$$\sum_\alpha \zeta_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p}) = m[\gamma^5\gamma_\mu g^\mu - \mathbb{I}]. \tag{32b}$$

The γ^μ, in the Weyl realization, are taken to be

$$\gamma^0 \equiv \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}, \qquad \gamma^i \equiv \begin{pmatrix} \mathbb{O} & -\sigma^i \\ \sigma^i & \mathbb{O} \end{pmatrix},$$
$$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}. \tag{33}$$

III. ELKO FERMIONIC FIELDS OF MASS-DIMENSION ONE: LAGRANGIAN DENSITIES

Confining to the preferred frame, we now examine the physical and mathematical content of two quantum fields [25]:

$$\Lambda(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\mathbf{p})}} \sum_\alpha [a_\alpha(\mathbf{p})\xi_\alpha(\mathbf{p})e^{-ip_\mu x^\mu} + b^\ddagger_\alpha(\mathbf{p})\zeta_\alpha(\mathbf{p})e^{+ip_\mu x^\mu}] \tag{34}$$

and

$$\lambda(x) \equiv \Lambda(x)|_{b^\ddagger(\mathbf{p})\to a^\ddagger(\mathbf{p})}. \tag{35}$$

We assume that the annihilation and creation operators satisfy the fermionic anticommutation relations [26]

$$\{a_\alpha(\mathbf{p}), a^\ddagger_{\alpha'}(\mathbf{p}')\} = (2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')\delta_{\alpha\alpha'}, \tag{36a}$$

$$\{a_\alpha(\mathbf{p}), a_{\alpha'}(\mathbf{p}')\} = 0, \qquad \{a^\ddagger_\alpha(\mathbf{p}), a^\ddagger_{\alpha'}(\mathbf{p}')\} = 0. \tag{36b}$$

Similar anticommutators are assumed for the $b_\alpha(\mathbf{p})$ and $b^\ddagger_\alpha(\mathbf{p})$. The adjoint field $\overset{\neg}{\Lambda}(x)$ is defined as

$$\overset{\neg}{\Lambda}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\mathbf{p})}} \sum_\alpha [a^\ddagger_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p})e^{+ip_\mu x^\mu} + b_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p})e^{-ip_\mu x^\mu}]. \tag{37}$$

The results contained in Eqs. (25a)–(25d) assure us that it is the Klein-Gordon, and not the Dirac, operator that annihilates the fields $\Lambda(x)$ and $\lambda(x)$. The associated Lagrangian densities are

$$\mathcal{L}^\Lambda(x) = \partial^\mu\overset{\neg}{\Lambda}(x)\partial_\mu\Lambda(x) - m^2\overset{\neg}{\Lambda}(x)\Lambda(x),$$
$$\mathcal{L}^\lambda(x) = \mathcal{L}^\Lambda(x)|_{\Lambda\to\lambda}. \tag{38}$$

The mass dimensionality of these Elko fields is thus one, and not 3/2.

The mass dimensionality of a field can also be deciphered from constructing the Feynman-Dyson propagator. This matter is discussed in App. A 2.

A. Identification of Elko with dark matter

These results open up an entirely new and unexpected possibility for the dark matter sector. The primary observations that suggest this are four-fold:

(1) Because of the mismatch in mass dimensionality of $\mathcal{D}_\Lambda = 1$ and $\mathcal{D}_\lambda = 1$ with the SM's matter fields $\mathcal{D}_\Psi = 3/2$, the new fermionic fields cannot enter the SM doublets.

(2) The Lagrangian densities associated with Elko fields do not carry the gauge symmetries of the SM. [See our remarks above Eq. (31).]

(3) The dimension four interactions of the $\Lambda(x)$ and $\lambda(x)$ with the standard model fields are restricted to those with the SM Higgs doublet $\phi(x)$. These are

$$\mathcal{L}^{\rm int}(x) = \phi^\dagger(x)\phi(x)[a_1\overset{\neg}{\Lambda}(x)\Lambda(x) + a_2\overset{\neg}{\lambda}(x)\lambda(x) + a_3(\overset{\neg}{\Lambda}(x)\lambda(x) + \overset{\neg}{\lambda}(x)\Lambda(x))],$$

where the a's are unknown coupling constants.

(4) By virtue of their mass dimensionality, the new dark matter fields are endowed with dimension four self-interactions,

$$\mathcal{L}^{\rm self}(x) = b_1(\overset{\neg}{\Lambda}(x)\Lambda(x))^2 + b_2(\overset{\neg}{\lambda}(x)\lambda(x))^2 + b_3[(\overset{\neg}{\Lambda}(x)\lambda(x))^2 + (\overset{\neg}{\lambda}(x)\Lambda(x))^2], \tag{39}$$

where the b's are unknown coupling constants. Observational evidence suggests that dark matter needs to be self-interacting [1–4,27].

Combined, the enumerated Elko properties not only render Elko dark with respect to the SM matter fields, but they also endow it with various observationally attractive properties. It is worth emphasizing that all of these properties are intrinsic to Elko, and arise in a natural way.

B. The locality structure of Elko

The canonically conjugate momenta to the Elko fields are

$$\Pi(x) = \frac{\partial \mathcal{L}^{\Lambda}}{\partial \dot{\Lambda}} = \frac{\partial}{\partial t}\overset{\neg}{\Lambda}(x), \tag{40}$$

and similarly $\pi(x) = \frac{\partial}{\partial t}\overset{\neg}{\lambda}(x)$. The calculational details for the two fields now differ significantly. We begin with the evaluation of the equal time anticommutator for the $\Lambda(x)$ and its conjugate momentum, and find

$$\begin{aligned}&\{\Lambda(\mathbf{x}, t), \Pi(\mathbf{x}', t)\}\\ &= i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2m}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\\ &\times \underbrace{\sum_{\alpha}[\xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(-\mathbf{p})\overset{\neg}{\zeta}_\alpha(-\mathbf{p})]}_{=2m[\mathbb{I}+\mathcal{G}(\mathbf{p})]}\end{aligned} \tag{41}$$

or, equivalently,

$$\{\Lambda(\mathbf{x}, t), \Pi(\mathbf{x}', t)\} = i\delta^3(\mathbf{x}-\mathbf{x}')\mathbb{I} + i\int \frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\mathbf{p}). \tag{42}$$

The anticommutators for the particle/antiparticle annihilation and creation operators suffice to yield the remaining locality conditions,

$$\{\Lambda(\mathbf{x}, t), \Lambda(\mathbf{x}', t)\} = \mathbb{O}, \qquad \{\Pi(\mathbf{x}, t), \Pi(\mathbf{x}', t)\} = \mathbb{O}. \tag{43}$$

Since the integral on the right-hand side of Eq. (42) vanishes only along the $\pm\hat{z}_e$ axis, the preferred axis also becomes the *axis of locality.*

For the equal time anticommutator of the $\lambda(x)$ field with its conjugate momentum, we find

$$\begin{aligned}\{\lambda(\mathbf{x}, t), \pi(\mathbf{x}', t)\} = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2m}\sum_{\alpha}[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}(\xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p})\\ - \zeta_\alpha(-\mathbf{p})\overset{\neg}{\zeta}_\alpha(-\mathbf{p}))],\end{aligned} \tag{44}$$

which, using the same argument as before, yields

$$\{\lambda(\mathbf{x}, t), \pi(\mathbf{x}', t)\} = i\delta^3(\mathbf{x}-\mathbf{x}')\mathbb{I} + i\int \frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\mathbf{p}). \tag{45}$$

The difference arises in the evaluation of the remaining anticommutators. The equal time λ-λ anticommutator reduces to

$$\begin{aligned}&\{\lambda(\mathbf{x}, t), \lambda(\mathbf{x}', t)\}\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\mathbf{p})}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\\ &\times \underbrace{\sum_{\alpha}[\xi_\alpha(\mathbf{p})\zeta^T_\alpha(\mathbf{p}) + \zeta_\alpha(-\mathbf{p})\xi^T_\alpha(-\mathbf{p})]}_{\equiv\Omega(\mathbf{p})}.\end{aligned} \tag{46}$$

Now using explicit expressions for $\xi_\alpha(\mathbf{p})$ and $\zeta_\alpha(\mathbf{p})$, we find that $\Omega(\mathbf{p})$ identically vanishes. Eq. (46) then implies

$$\{\lambda(\mathbf{x}, t), \lambda(\mathbf{x}', t)\} = \mathbb{O}. \tag{47}$$

Finally, the equal time π-π anticommutator simplifies to

$$\begin{aligned}&\{\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)\}\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{E(\mathbf{p})}{2m}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\\ &\times \underbrace{\sum_{\alpha}[(\overset{\neg}{\xi}_\alpha(\mathbf{p}))^T\overset{\neg}{\zeta}_\alpha(\mathbf{p}) + (\overset{\neg}{\zeta}_\alpha(-\mathbf{p}))^T\overset{\neg}{\xi}_\alpha(-\mathbf{p})]}_{=\mathbb{O},\ \text{by a direct evaluation}},\end{aligned}$$

yielding

$$\{\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)\} = \mathbb{O}. \tag{48}$$

Eqs. (42), (43), and (45)–(48) establish that $\Lambda(x)$ and $\lambda(x)$ are *local* quantum fields along the preferred axis, $\hat{z}_e$. We propose to call $\hat{z}_e$ the *axis of locality* in the dark sector.

IV. CONCLUDING REMARKS

Modulo its specialization to the Majorana field, Weinberg's monographic work [8] establishes the uniqueness of the Dirac quantum field for spin 1/2 particles. Seen from that perspective the Ahluwalia-Grumiller work on Elko in 2005 was unexpected. Elko found significant interest among mathematical physicists and cosmologists [28–41]. In these papers one dealt with Elko as spinors and not as a quantum field. Hence, no contradiction with Weinberg's theoremlike work occurred. Gillard and Martin showed that if Elko were to be taken as "good" quantum fields, Poincaré symmetries would be violated in some form or the other [42]. The results presented in this communication explicitly confirm this and show that the violation occurs in a rather subtle way. Despite this, Elko stands as a natural dark matter candidate. Its darkness with respect to the SM matter and gauge fields follows immediately from its intrinsic mass dimensionality. It admits an unsuppressed quartic self coupling. Additionally, it points towards the existence of a preferred axis, along which the Elko quantum fields enjoy locality. Although Elko is nonlocal when the frame is not aligned to the preferred axis, Fabbri [41] has shown that the fields do not violate causality in the sense of Velo and Zwanziger [43].

Recent results seem to suggest that the Elko quantum fields satisfy the symmetry of very special relativity (VSR) proposed by Cohen and Glashow [44]. The HOM(2) and SIM(2) VSR groups naturally incorporate a preferred axis which may be identified with the axis of locality. This will be published in a forthcoming paper.

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APPENDIX A

1. Dirac spin sums and a "misleading" derivation of the Dirac equation

With a minor departure from the historical path, the Dirac counterpart of Eqs. (32a) and (32b) may be constructed as follows. Instead of (6), we start with

$$\psi_D \equiv \begin{pmatrix} \phi_r \\ \phi_\ell \end{pmatrix}. \tag{A1}$$

The helicities of ϕ_r and ϕ_ℓ are identical and are determined by requiring that ψ_D be eigenspinors of the parity operator $S(P)$. Again, there are four independent rest spinors. (These differ from those mentioned in Sec. I A only in that we now work in the "helicity basis.")

$$u_{+1/2}(\boldsymbol{\epsilon}) = \begin{pmatrix} \phi_r^+(\boldsymbol{\epsilon}) \\ \phi_\ell^+(\boldsymbol{\epsilon}) \end{pmatrix}, \qquad u_{-1/2}(\boldsymbol{\epsilon}) = \begin{pmatrix} \phi_r^-(\boldsymbol{\epsilon}) \\ \phi_\ell^-(\boldsymbol{\epsilon}) \end{pmatrix}, \tag{A2}$$

$$v_{+1/2}(\boldsymbol{\epsilon}) = \begin{pmatrix} \phi_r^-(\boldsymbol{\epsilon}) \\ -\phi_\ell^-(\boldsymbol{\epsilon}) \end{pmatrix}, \qquad v_{-1/2}(\boldsymbol{\epsilon}) = \begin{pmatrix} -\phi_r^+(\boldsymbol{\epsilon}) \\ \phi_\ell^+(\boldsymbol{\epsilon}) \end{pmatrix}. \tag{A3}$$

The $u(\mathbf{p})$ and $v(\mathbf{p})$ for an arbitrary momentum are obtained via the action of the boost κ:

$$u(\mathbf{p}) = \kappa u(\boldsymbol{\epsilon}), \qquad v(\mathbf{p}) = \kappa v(\boldsymbol{\epsilon}). \tag{A4}$$

These lead to the spin sums

$$\sum_\beta u_\beta(\mathbf{p})\bar{u}_\beta(\mathbf{p}) = m\left[\frac{\gamma_\mu p^\mu}{m} + \mathbb{I}\right], \tag{A5a}$$

$$\sum_\beta v_\beta(\mathbf{p})\bar{v}_\beta(\mathbf{p}) = m\left[\frac{\gamma_\mu p^\mu}{m} - \mathbb{I}\right], \tag{A5b}$$

where β takes two values: $+1/2$ and $-1/2$. As before, the right-hand sides in the above expression simply express the result of a direct evaluation of the left-hand sides. These are covariant.

We thus see that in the Dirac construct (whether it be at the level of spinors or at the level of a quantum field), no preferred frame is introduced. For Majorana spinors, and Elko, the conclusion is both unexpected and inevitable. This difference—as pertaining to the existence of a preferred frame—between the Dirac and Majorana spinors, along with their cousins Elko, to our knowledge is completely unknown. This conclusion carries distinct echoes of the unpublished notes [45] which eventually, in collaboration with Grumiller, led to the discovery reported in Refs. [5,6].

If we multiply Eq. (A5a) by $u_{\beta'}(\mathbf{p})$ from the right, and use $\bar{u}_\beta(\mathbf{p})u_{\beta'}(\mathbf{p}) = 2m\delta_{\beta\beta'}$, and carry out a similar exercise with Eq. (A5b), then after a minor rearranging we obtain

$$(\gamma_\mu p^\mu - m\mathbb{I})u(\mathbf{p}) = 0, \tag{A6}$$

$$(\gamma_\mu p^\mu + m\mathbb{I})v(\mathbf{p}) = 0. \tag{A7}$$

These are indeed Dirac equations in momentum space. With $p^\mu \to i\partial^\mu$ and

$$\psi(x) \equiv \begin{cases} u(\mathbf{p}) & \exp(-ip_\mu x^\mu) \\ v(\mathbf{p}) & \exp(+ip_\mu x^\mu), \end{cases} \tag{A8}$$

these yield the well-known Dirac equation in the configuration space

$$(i\gamma_\mu\partial^\mu - m\mathbb{I})\psi(x) = 0. \tag{A9}$$

To associate these with the dynamics of spin 1/2 spinors, particularly in the context of quantum field theory [where $\psi(x)$ is elevated to a spinor field $\Psi(x)$] requires that, in addition, the vacuum expectation value, $\langle|\mathcal{T}[\Psi(x')\bar{\Psi}(x)]|\rangle$, be proportional to the relevant Green's function. That is to say, it is not sufficient to find an operator, such as $(i\gamma_\mu\partial^\mu - m\mathbb{I})$, or the Klein-Gordon operator, that annihilates $\Psi(x)$ for it to serve in the Lagrangian density of the field $\Psi(x)$. It must also satisfy the said requirement. This will become abundantly clear from what follows in the context of Elko.

While we do consider the above "derivation" of the Dirac equation misleading, it does serve to tell us that the Dirac spinors are eigenspinors of $\gamma_\mu p_\mu$ with eigenvalues $\pm m$:

$$\gamma_\mu p^\mu u(\mathbf{p}) = +mu(\mathbf{p}), \qquad \gamma_\mu p^\mu v(\mathbf{p}) = -mv(\mathbf{p}). \tag{A10}$$

The Elko counterpart is

$$\mathcal{G}(\mathbf{p})\xi(\mathbf{p}) = +\xi(\mathbf{p}), \qquad \mathcal{G}(\mathbf{p})\zeta(\mathbf{p}) = -\zeta(\mathbf{p}). \tag{A11}$$

It again emphasizes that identities such as these should not be mistaken for dynamical equations. In particular, $\mathcal{G}(\mathbf{p})$, unlike its Dirac counterpart $\gamma_\mu p^\mu$, contains no time derivative.

2. Elko time ordering and propagators

The mass dimensionality of a field can also be deciphered from constructing the Feynman-Dyson propagator. This involves defining a time-ordering operator. The existence of a preferred direction, however, raises questions with regard to the definition in the context of Elko. In what follows, we first adopt the standard definition of the fermionic time-ordering operator, and then we invoke a consistency argument to formulate a redefinition for Elko.

Let $\mathcal{T}$ be the standard fermionic time-ordering operator. Then, a straightforward calculation yields

$$\langle|\mathcal{T}[\Lambda(x')\overset{\neg}{\Lambda}(x)]|\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\mathbf{p})} \times \sum_\alpha [\theta(t'-t)\xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p})e^{-ip_\mu(x'^\mu - x^\mu)} - \theta(t-t')\zeta_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p})e^{+ip_\mu(x'^\mu - x^\mu)}], \tag{A12}$$

where the step function $\theta(t)$ equals unity for $t > 0$ and vanishes for $t < 0$.

Using the spin sums (27a) and (27b), setting $q^\mu = (q^0, \mathbf{q} = \mathbf{p})$, and using the standard integral representation for the $\theta(t)$, Eq. (A12) simplifies to

$$\langle|\mathcal{T}[\Lambda(x')\overset{\neg}{\Lambda}(x)]|\rangle = i\int \frac{d^4q}{(2\pi)^4} e^{-iq_\mu(x'^\mu - x^\mu)}\left[\frac{\mathbb{I} + \mathcal{G}(\mathbf{q})}{q_\mu q^\mu - m^2 + i\epsilon}\right], \tag{A13}$$

where the limit $\epsilon \longrightarrow 0^+$ is understood [46]. If there were no preferred axis, then the integral involving the $\mathcal{G}(\mathbf{q})$ term would have identically vanished. Consistency with result (38) suggests that, in Elko quantum field theory, one may need to modify the definition of the time-ordered product to $\mathcal{T}_\#$, such that

$$\langle|\mathcal{T}_\#[\Lambda(x')\overset{\neg}{\Lambda}(x)]|\rangle = i\int \frac{d^4q}{(2\pi)^4} e^{-iq_\mu(x'^\mu - x^\mu)}\left[\frac{\mathbb{I}}{q_\mu q^\mu - m^2 + i\epsilon}\right]. \tag{A14}$$

To decipher the mass dimensionality, let $\mathcal{D}_\Lambda$ be the mass dimensionality of $\Lambda(x)$. Then the left-hand side of the above equation has mass dimension $2\mathcal{D}_\Lambda$. As for the right-hand side, the mass dimensionality is 2. This gives $\mathcal{D}_\Lambda = 1$. Similarly, a simple computation shows that $\langle|\mathcal{T}_\#[\Lambda(x')\overset{\neg}{\Lambda}(x)]|\rangle = \langle|\mathcal{T}_\#[\lambda(x')\overset{\neg}{\lambda}(x)]|\rangle$. As such, $\mathcal{D}_\lambda = 1$.

Applying the operator $[\partial'^\mu\partial'_\mu + m^2]$ from the left on both sides of Eq. (A14) gives

$$[\partial'^\mu\partial'_\mu + m^2]\langle|\mathcal{T}_\#[\Lambda(x')\overset{\neg}{\Lambda}(x)]|\rangle = -i\delta^4(x'^\mu - x^\mu). \tag{A15}$$

In comparison, for the Dirac field,

$$\langle|\mathcal{T}[\Psi(x')\bar{\Psi}(x)]|\rangle = i\int \frac{d^4q}{(2\pi)^4} e^{-iq_\mu(x'^\mu - x^\mu)}\left[\frac{m\mathbb{I} + \gamma^\mu q_\mu}{q_\mu q^\mu - m^2 + i\epsilon}\right]. \tag{A16}$$

This well-known result gives $\mathcal{D}_\Psi = \frac{3}{2}$. The reader is reminded that the $\gamma^\mu q_\mu$ structure appears here through the spin sums which, in the logical framework of this communication, do not invoke any wave equation or a Lagrangian density. Applying the operator $[i\gamma^\mu\partial'_\mu - m]$ from the left on both sides of Eq. (A16) yields

$$[i\gamma^\mu\partial'_\mu - m]\langle|\mathcal{T}[\Psi(x')\bar{\Psi}(x)]|\rangle = i\delta^4(x'^\mu - x^\mu). \tag{A17}$$

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[26] Here we use ‡ for the creation operator instead of † to allow for the possibility that the state space for Elko may be different from the Dirac particle.

[27] We parenthetically remark that the interactions with the standard model gauge fields—with $F^{\rm SM}_{\mu\nu}(x)$ symbolically representing the associated field strength tensors—through Pauli terms

$$\mathcal{L}^{\rm Pauli}(x) = \overset{\neg}{\Lambda}(x)[\gamma^\mu, \gamma^\nu]\lambda(x)F^{\rm SM}_{\mu\nu}(x), \quad \text{etc.}$$

may exist in principle. However, we consider them to have vanishing coupling strength, as $\mathcal{L}^{\Lambda}(x)$ and $\mathcal{L}^{\lambda}(x)$ do not carry invariance under SM gauge transformations.

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